RESPONSE OF FIRST-ORDER SYSTEMS

TRANSFER FUNCTION

MERCURY THERMOMETER

The transfer function for a first order system by considering the unsteady-state behavior of an ordinary mercury-in-glass thermometer is derived.

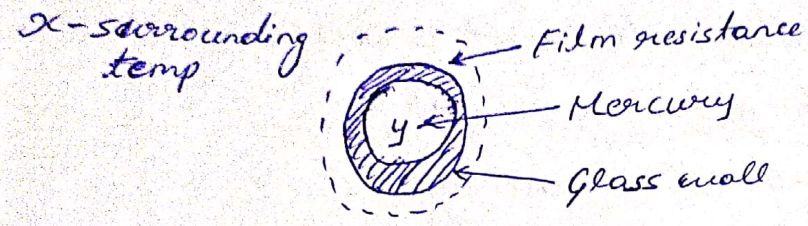


Fig - C/s view of thermometer

Consider the thermometer to be located in a flowing stream of fluid for which the temp. 'x' varies with time. The response or the time variation of the thermometer reading 'y' for a particular change in x is to be found.

The assumptions used for this analysis are:

1. All the resistance to heat transfer resides in the film surrounding the bulb (i.e. the resistance offered by the glass and mercury is neglected).

2. All the thermal capacity is in the mercury. Furthermore, at any instant the mercury assumes a uniform temp. throughout.

3. The glass wall containing the mercury does not expand or contract during the transient response. [In an actual thermometer, the expansion of the wall has an additional effect on the response of the thermometer reading].

It is assumed that the thermometer is initially at steady state. i.e. Before time zero, there is no change in temp with time. At time zero the thermometer will be subjected to some change in the surrounding temp. $x(t)$.

By applying the unsteady-state energy balance,

Input rate = Output rate - rate of accumulation

Input rate - output rate = rate of accumulation

$$(hAx - hAy) - 0 = mc\frac{dy}{dt}$$

$$hA(x-y) - 0 = mc\frac{dy}{dt} \quad \text{— (1)}$$

where,

A - Surface area of bulb for heat transfer, m^2

C - Heat capacity of mercury, J/kg.K.

m - mass of mercury in bulb, kg

t - time, hr

h - film co-efficient of heat transfer, $J/s.m^2.K$

Eqn. (1) states that the rate of flow of heat through the film resistance surrounding the bulb causes the internal energy of the mercury to increase at the same rate. The increase in internal energy is manifested by a change in temperature and a corresponding expansion of mercury, which causes the mercury column, or 'reading' of the thermometer, to rise

The co-efficient 'h' will depend on the flow rate and properties of the surrounding fluid and the dimensions of the bulb. Assume that h is constant for a particular installation of the thermometer.

Analysis has resulted in Eq (1), which is a first-order differential eqn. Before solving the eqn. by Laplace transform, deviation variables will be introduced into eqn ①.

Prior to the change in x occurs, the thermometer is at s.s and the derivative dy/dt is zero.

For s.s condition, eqn ①

$$hA(x_s - y_s) = 0, \quad t < 0 \quad —② $$

Subscript 's' is used to indicate that the variable is s.s. value

Eqn ② states that the thermometer reads true, bath temp.

i.e. $y_s = x_s$

Subtracting eqn ② from eqn ①. gives

$$hA[(x-x_s) - (y-y_s)] = m.C \frac{d(y-y_s)}{dt} \quad —③$$

$$\frac{d(y-y_s)}{dt} = \frac{dy}{dt}$$ because y_s is a constant.

The deviation variables are defined to be the differences between the variables and their s.s. values.

$$X = x - x_s$$

$$Y = y - y_s$$

Eqn ③ becomes.

$$hA[X - Y] = mC\frac{dY}{dt} \quad —④$$

Let $\frac{mC}{hA} = \tau$. then eq ④ is

$$X - Y = \tau \frac{dY}{dt} \quad —⑤$$

Taking the Laplace transform of eqn ⑤ gives

$$X(s) - Y(s) = \tau s Y(s) \quad —⑥$$

Rearranging eqn ⑥ as a ratio of $Y(s)$ to $X(s)$ gives.

$$Y(s)[T's+1] = X(s)$$

$$\frac{Y(s)}{X(s)} = \frac{1}{T's+1} \quad \text{——— (7)}$$

T' :- is called the time constant of the system and has the units of time. i.e seconds.

The experession on the right side of Eq. (7) is called the transfer function of the system. It is the ratio of the Laplace transform of the deviation in thermometer. reading to the Laplace transform of the deviation in the surrounding temp.

First Order System.-

Any physical system for which the relation between L.T of input and output deviation variables is of the form given as $\frac{Y(s)}{X(s)} = \frac{1}{T's+1}$ is called 1st order system. It is also known as first-order lag & single exponential stage.

TRANSFER FUNCTION : A transfer function relates two variables in a physical process, one is cause (forcing function OR input variable) and the other is the effect (response OR output variable).

Eg:- In mercury thermometer, the surrounding temp. is the cause OR input, whereas the thermometer reading is the effect OR output.

$$\text{Transfer function} = G(s) = \frac{Y(s)}{X(s)}$$

where, $G(s)$ = symbol for transfer function.

$X(s)$ = transform of forcing function OR input in deviation form

$Y(s)$ – transform of response OR output, in deviation form

The transfer function completely describes the dynamic characteristics of the system.

For a particular input variation $X(t)$ for which the transform is $X(s)$, the response of the system is

$$Y(s) = G(s) \cdot X(s)$$

By taking the inverse of $Y(s)$, we get $Y(t)$, the response of system.

The transfer function results from a linear differential equation, therefore, the principle of superposition is applicable. This means that the transformed response of a system with transfer function $G(s)$ to a forcing function.

If $X(s) = a_1 X_1(s) + a_2 X_2(s)$

where,

X_1, X_2 – particular functions

a_1, a_2 – constants

then, $Y(s) = G(s)\, X(s)$

$$= a_1 G(s) X_1(s) + a_2 G(s) X_2(s)$$

$$= a_1 Y_1(s) + a_2 Y_2(s)$$

$Y_1(s)$ and $Y_2(s)$ are the responses to X_1 and X_2 alone, respectively.

Ex – The response of the mercury thermometer to a sudden change in surrounding temp. of 10 °F is simply twice the response to a sudden change of 5 °F in surrounding temp.

Block diagram

$$X(s) \longrightarrow \boxed{G(s)} \longrightarrow Y(s)$$

The functional relationship contained in a transfer function is often expressed by a block-diagram. as shown in fig.

Arrow entering — is forcing function (or) input variable
Arrow leaving — is response (or) output variable.
Inside the box — is placed the transfer function

The transfer function G(s) in the box "operates" on the input function X(s) to produce an output function Y(s).

Types of forcing functions:

1. Step
2. Impulse.
3. Sinusoidal

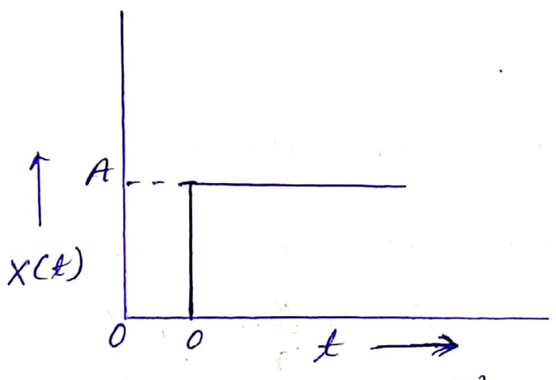
Forcing functions:

1. step function – Mathematically, the step function of magnitude A can be expressed as,

$$X(t) = A\,u(t)$$

where, u(t) — is the unit-step function.

Graphical representation



$X = 0,\ t < 0$

$X = A,\ t \geq 0$

$X(s) = \frac{A}{s}$ – Transform fun.

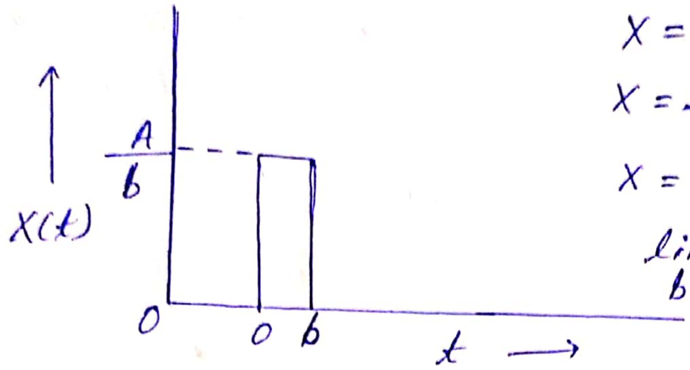
Ex - step change in flow rate by sudden opening of a value.

2. Impulse function – Mathematically, the impulse function of magnitude A is defined as,

$$X(t) = A\,\delta(t)$$

where, $\delta(t)$ – is the unit-impulse function.

Graphical representation.-



$X = 0, \; t < 0$

$X = \frac{A}{b}, \; 0 \leq t \leq b$

$X = 0, \; t > b$

$\lim_{b \to 0} X(t) = A \cdot \delta(t)$

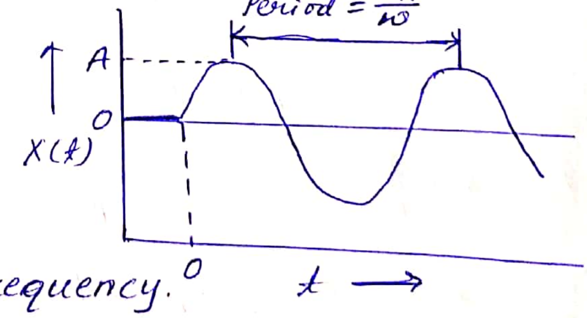
$L\{A\delta(t)\} = A$

The true impulse function, obtained by letting $b \downarrow 0$ in fig. has Laplace transform A. It is used more frequently as a mathematical aid than as an actual input to a physical system. For some system it is difficult even to approximate an impulse forcing function. For this reason the representation of fig. is valuable, since this form can usually be approximated physically by application and removal of a step function. If the time duration b is sufficiently small the forcing function gives a response that closely resembles the response to a true impulse.

Sinusoidal input - This function is represented mathematically by the equations,

$X = 0 \qquad t < 0$

$X = A \sin \omega t \quad t \geq 0$



where, A is the amplitude

ω - is the radian frequency.

The radian frequency ω is related to the frequency f in cycles per unit time by $\omega = 2\pi f$.

The transform is $X(s) = \frac{A\omega}{(s^2 + \omega^2)}$.

$X = 0, \; t < 0$

$X = A \sin \omega t, \; t \geq 0$

$X(s) = \frac{A\omega}{s^2 + \omega^2}$

This forcing function forms the basis of an important branch of control theory known as "frequency response".

STEP RESPONSE FOR FIRST ORDER SYSTEM.

If a step change of magnitude A is introduced into a first-order system, the transform of $X(t)$ is,

$$X(s) = \frac{A}{s} \quad \text{——} \quad ①$$

WKT, the transform function for I order system,

$$\frac{Y(s)}{X(s)} = \frac{1}{\tau s+1} \quad \text{——} \quad ②$$

Combining eqs. ① & ② gives,

$$Y(s) = \frac{A}{s} \cdot \frac{1}{\tau s+1} \quad \text{——} \quad ③$$

on expansion by partial fractions,

$$Y(s) = \frac{A/\tau}{(s)\left(s+\frac{1}{\tau}\right)} = \frac{C_1}{s} + \frac{C_2}{s+\frac{1}{\tau}} \quad \text{——} \quad ④$$

Solving for the constants C_1 and C_2.

$$\frac{A/\tau}{(s)\left(s+\frac{1}{\tau}\right)} = \frac{C_1}{s} + \frac{C_2}{s+\frac{1}{\tau}}$$

$$\frac{A}{\tau} = C_1\left(s+\frac{1}{\tau}\right) + C_2 s.$$

Put $s=0$, $\quad \frac{A}{\tau} = C_1\left(\frac{1}{\tau}\right)$

$$C_1 = A$$

Put $\left(s+\frac{1}{\tau}\right)=0$, $\quad s = -\frac{1}{\tau}$

$$\frac{A}{\tau} = C_1(0) + C_2\left(-\frac{1}{\tau}\right)$$

$$C_2 = -A$$

$$Y(s) = \frac{A}{s} - \frac{A}{s+\frac{1}{\tau}}$$

Taking the inverse transform gives the time response for Y:

$$Y(t) = 0 \qquad t < 0$$

$$Y(t) = A(1 - e^{-t/\tau}) \quad t \geq 0 \qquad \text{--- (5)}$$

The response is zero before $t = 0$.
Eqn (5) is plotted in terms of the dimensionless quantities $\frac{Y(t)}{A}$ and $\frac{t}{\tau}$.

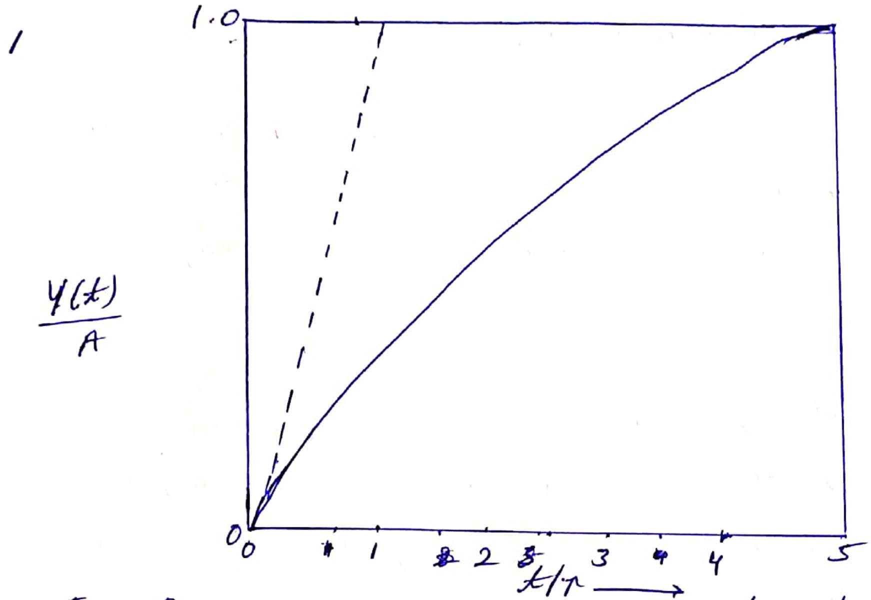


Fig - Response of a first-order system to a step input

Ex. - Immediately after the thermometer is placed in the new environment, the temp. difference between the mercury in the bulb and the bath temp is at its maximum value. With simple lumped-parameter model, should expect the flow of heat to commence immediately, with the result that the mercury temp. rises, causing a corresponding rise in the column of mercury. As the mercury temp. rises, the driving force causing heat to flow into the mercury will diminish, with the result that the mercury temp. changes at a slower rate as time proceeds. This description of the response based on physical grounds does agree with the response given by eqn (5) and shown graphically as in fig.

Some features of step response are :

1. The value of $Y(t)$ reaches 63.2% of its ultimate value when the time elapsed is equal to one time constant τ. When the time elapsed is 2τ, 3τ, and 4τ, the percent response is 86.5, 95 and 98 respectively.
From these facts, the response is essentially completed in three to four time constants.

2. From eqn. (5) that the slope of the response curve at the origin is 1. This means that, if the initial rate of change of $Y(t)$ were maintained, the response would be complete in one time constant.

3. A consequence of the principle of superposition is that the response to a step input of any magnitude A may be obtained directly from fig. by multiplying the ordinate by A. Fig. actually gives the response to a unit-step function input, from which all other step responses are derived by superposition.

Example –

1. A thermometer having a time constant of 0.1 min is at a steady-state temp. of 90°F. At time $t=0$, the thermometer is placed in a temp. bath maintained at 100°F. Determine the time needed for the thermometer to read 98°.

solution –

Data given, $\tau = 0.1$ min

$x_s = 90°$

$A = 10°$

The ultimate thermometer reading = 100°

ultimate value of the deviation variable $Y(\infty)$ is 10°.

When the thermometer reads 98°, $Y(t) = 8°$.

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WKT, $Y(t) = A(1-e^{-t/\tau})$

$$8 = 10(1-e^{-t/0.1})$$

$$t = 0.161 \text{ min}$$

By refering to the fig.

$$\frac{Y}{A} = 0.8 \text{ at } \frac{t}{\tau} = 1.6$$

IMPULSE RESPONSE:

The Laplace transform of a unit impulse is,

$$X(s) = 1$$

WKT T.F of a 1st order system

$$\frac{Y(s)}{X(s)} = \frac{1}{\tau s+1}$$

$$\therefore \quad Y(s) = \frac{1}{\tau s+1} = \frac{1/\tau}{s+1/\tau}$$

The inverse of Y(s) from the table of transform

$$\tau Y(t) = e^{-t/\tau}$$

A plot of this response in terms of the variable t/τ and $\tau Y(t)$, is shown above.

The response to an impulse of magnitude A is obtained, as usual, by multiplying $\tau Y(t)$ from fig. by A/τ.

The response rises immediately to 1.0 and then decays exponentially. Such an abrupt rise is physically impossible, but it is approached by the response to a finite pulse of narrow width.

SINUSOIDAL RESPONSE :

Ex- Consider a mercury thermometer to be in equilibrium with a temp. bath at temp x_s. At some time $t=0$, the bath temp. begins to vary according to the relationship.

$$x = x_s + A \sin \omega t \qquad t>0 \quad — ①$$

where,
x - temp. of bath
x_s - temp. of bath before sinusoidal disturbance is applied
A - amplitude of variation in temp.
ω - radian frequency, rad/time

Introducing the deviation variables X as,

$$X = x - x_s \quad —— ②$$

Substituting ② in ①

$$x - x_s = A \sin \omega t$$

$$X = A \sin \omega t \quad —— ③$$

The L.T of eqn. ③ is

$$X(s) = \frac{A\omega}{s^2+\omega^2} \quad —— ④$$

W.K.T. L.T of 1st order system is.

$$\frac{Y(s)}{X(s)} = \frac{1}{\tau s + 1}$$

$$Y(s) = \frac{A\omega}{s^2+\omega^2} \cdot \frac{1/\tau}{s+1/\tau} \quad ⑤$$

Eqn. ⑤ can be solved for $Y(t)$ by means of a partial-fraction expansion,

$$\frac{A\omega}{(s^2+\omega^2)} \cdot \frac{1/\tau}{(s+1/\tau)} = \frac{C_1}{(s+j\omega)} + \frac{C_2}{(s-j\omega)} + \frac{C_3}{(s+1/\tau)}$$

Solving by partial fraction, result is

$$Y(t) = \frac{A\omega\tau e^{-t/\tau}}{\tau^2\omega^2+1} - \frac{A\omega\tau}{\tau^2\omega^2+1}\cos\omega t + \frac{A}{\tau^2\omega^2+1}\sin\omega t \quad — (6)$$

Using trigonometric identity eqn (6) can be written as,

$$p\cos A + q\sin A = r\sin(A+\theta) \quad — (7)$$

where, $r = \sqrt{p^2+q^2}$ $\tan\theta = \frac{p}{q}$

Applying the identity of eqn. (7) & (6) gives

$$Y(t) = \frac{A\omega\tau}{\tau^2\omega^2+1} e^{-t/\tau} + \frac{A}{\sqrt{\tau^2\omega^2+1}}\sin(\omega t+\phi) \quad — (8)$$

where, $\phi = \tan^{-1}(-\omega\tau)$

As $t \to \infty$, the 1st term on the R.H.S of eq (8) vanishes and leaves only the ultimate periodic solution, which is sometimes called the s.s. solution,

$$Y(t)\Big|_s = \frac{A}{\sqrt{\tau^2\omega^2+1}}\sin(\omega t+\phi) \quad — (9)$$

By comparing eq. $X = A\sin\omega t$ ~~and eq. (9)~~ for the input forcing function with eq. (9) for the ultimate periodic response,

1. The output is a sine wave with a frequency ω equal to that of the input signal.
2. The ratio of output amplitude to input amplitude is $1/\sqrt{\tau^2\omega^2+1}$. This is always smaller than 1. This is known as signal is attenuated (or) attenuated signal.
3. The output lags behind the input by an angle $|\phi|$. It is clear that lag occurs, for the sign of ϕ is always negative.

$\phi < 0$, phase lag

$\phi > 0$, phase lead

For a particular system for which the time constant τ is a fixed quantity, it is from eq. (9) that the attenuation of amplitude and the phase angle ϕ depends only on the frequency ω.

The attenuation and phase lag increase with frequency. but the phase lag can never exceed $90°$ and approaches this value asymptotically.

Ex – A mercury thermometer having a time constant of 0.1 min is placed in a temp bath at 100°F and allowed to come to equilibrium with the bath. At time $t=0$, the temp. of the bath begins to vary sinusoidally about its average temp. of 100°F with an amplitude of 2°F. If the frequency of oscillation is $10/\pi$ cycles/min, plot the ultimate response of the thermometer reading as a function of time. What is the phase lag?

Solution

Data given, $\tau = 0.1$ min

$x_s = 100°F$

$A = 2°F$

$f = \frac{10}{\pi}$ cycles/min

$\omega = 2\pi f = 2\pi \frac{10}{\pi} = 20$ rad/min

From eq. (9) the amplitude of the response and the phase angle are calculated.

W.K.T. $Y(t)\Big|_s = \frac{A}{\sqrt{\tau^2\omega^2+1}} \sin(\omega t+\phi)$

$$\frac{A}{\sqrt{\tau^2\omega^2+1}} = \frac{2}{\sqrt{4+1}} = 0.896°F$$

$$\phi = -\tan^{-1}2 = -63.5°$$

(or) phase lag $= 63.5°$

The response of the thermometer is,

$\therefore\ Y(t) = 0.896 \sin(20t - 63.5°)$

Or

$Y(t) = 100 + 0.896 \sin(20t - 63.5°)$

To obtain the lag in terms rather than angle.

A frequency of $10/\pi$ cycles/min. means that a complete cycle (peak to peak) occurs in $\left(\frac{10}{\pi}\right)^{-1}$ min. Since one cycle is equivalent to 360° and the lag is 63.5°, the time corresponding to this lag is

$$\frac{63.5}{360} \times (\text{time for 1 cycle})$$

Or

$$Lag = \frac{63.5}{360} \cdot \frac{\pi}{10} = 0.0555 \text{ min}$$

In general, the lag in units of time is given by,

$$Lag = \frac{|\phi|}{360 f}.$$

when ϕ is expressed in degrees.

The response of the thermometer reading and the variation in bath temp are shown in fig.

It should be noted that the response shown in fig. holds only after sufficient time has elapsed for the non-periodic term of eqn. 8 to become negligible. For all practical purposes this term becomes negligible after a time equal to about 3τ.

If the response were desired beginning from the time the bath temp. begins to oscillate, it would be necessary to plot the complete response as given by eq. 8.

PHYSICAL EXAMPLES OF FIRST-ORDER SYSTEMS

Linearisation - A method for approximating the dynamic response of a non-linear system by a linear response is known as "Linearisation."

Examples of first-order systems

Liquid Level.

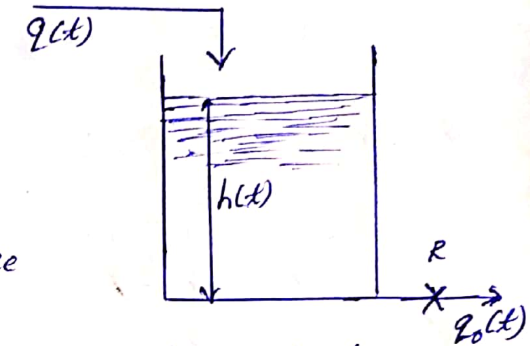

Fig-Liquid-level system

Consider the system as shown in fig. It consists of a tank of uniform C/s area 'A' to which is attached a flow resistance R such as a valve, a pipe or a weir.

Assume that q_o, the volumetric flow rate (volume/time) through the resistance, is related to the head h by the linear relationship.

$$q_o = \frac{h}{R} \quad \text{--- (1)}$$

A resistance that has this linear relationship between flow and head is referred to as a linear resistance.

A time-varying volumetric flow q of liquid of constant density ρ enters the tank.

Mass flow in - mass flow out = rate of accumulation of mass in the tank

$$\rho q(t) - \rho q_o(t) = \frac{d(\rho A h)}{dt}$$

$$q(t) - q_o(t) = A\frac{dh}{dt} \quad \text{--- (2)}$$

Combining eqn. (1) & (2) to eliminate $q_o(t)$ gives the following linear differential eqn.

$$q - \frac{h}{R} = A\frac{dh}{dt} \quad \text{--- (3)}$$

Introducing the deviation variables.

$\frac{dh}{dt} = 0$ (Initially the process is operating at s.s)

$$q_s - \frac{h_s}{R} = 0 \quad \text{——— (4)}$$

Subtracting eq (4) from (3) gives,

$$(q - q_s) = \frac{1}{R}(h - h_s) + A\frac{d(h-h_s)}{dt} \quad \text{——— (5)}$$

Defining the deviation variables are,

$$Q = q - q_s$$
$$H = h - h_s$$

Eq. (5) can be written as,

$$Q = \frac{1}{R}(H) + A\frac{dH}{dt} \quad \text{——— (6)}$$

Taking the transform.

$$Q(s) = \frac{1}{R}H(s) + A[sH(s) - H(0)]$$

$$Q(s) = \frac{1}{R}H(s) + AsH(s) \quad \text{——— (7)} \quad \because H(0) = 0$$

on rearranging into std. form of first order lag.

$$Q(s) = H(s)\left(\frac{1}{R} + A(s)\right) = H(s)\left[\frac{1 + RAS}{R}\right]$$

$$\frac{H(s)}{Q(s)} = \frac{R}{\tau s + 1} \quad \text{——— (8)} \quad [\tau = A.R]$$

The term 'R' is simply the conversion factor that relates $h(t)$ to $q(t)$ when the system is at steady state. For this reason, a factor K in the transfer function $\frac{K}{(\tau s + 1)}$ is often called the steady-state gain.

If flow rate $Q(t)$ changes according to a unit-step change.

$$Q(t) = u(t)$$

$u(t)$ – unit step change.

$$Q(s) = \frac{1}{s}$$

Combining this forcing function with eq. 8

$$H(s) = \frac{1}{s} \cdot \frac{R}{\tau s + 1}$$

Applying the final value theorem to $H(s)$

$$H(t)\Big|_{t\to\infty} = \lim_{s\to 0} [sH(s)] = \lim_{s\to 0} \frac{R}{\tau s + 1} = R.$$

This shows that the ultimate change in $H(t)$ for a unit change in $Q(t)$ is simply R.

If the transfer function relating the inlet flow $q(t)$ to the outlet flow is ~~simply R~~ desired.

then,
$$q_{os} = \frac{h_s}{R} \quad \text{—— (9)}$$

Subtracting eq. (9) from eq. (1) and using the deviation variable. $Q_o = q_o - q_{os}$ gives,

$$Q_o = \frac{H}{R} \quad \text{—— (10)}$$

Taking the transform of eqn. (10) gives,

$$Q_o(s) = \frac{H(s)}{R} \quad \text{—— (11)}$$

Combining eqn. (11) and (8) to eliminate $H(s)$ gives.

$$\frac{Q_o(s) \cdot R}{Q(s)} = \frac{R}{\tau s + 1}$$

$$\frac{Q_o(s)}{Q(s)} = \frac{1}{\tau s + 1} \quad \text{—— (12)}$$

The steady-state gain for this transfer function is dimensionless, which is to be expected because the input variable $q(t)$ and the output variable $q_o(t)$ have the same units (volume/time).

The possibility of approximating an impulse forcing function in the flow rate to the liquid-level system is quite real. The unit-impulse function is defined as a pulse of unit area as the duration of the pulse approaches zero, the impulse function can be approximated by suddenly increasing the flow to a large value for a very short time, i.e. we may pour very quickly a volume of liquid into the tank.

Ex- A tank having a time constant of 1 min. and a resistance of $\frac{1}{9}$ ft/cfm is operating at steady state with an inlet flow of 10 ft^3/min. At time $t=0$, the flow is suddenly increased to 100 ft^3/min for 0.1 min by adding an additional 9 ft^3 of water to the tank uniformly over a period of 0.1 min. Plot the response in tank level and compare with the impulse response.

solution -

The transform function of the process is.

$$\frac{H(s)}{Q(s)} = \frac{1}{9} \; \frac{1}{s+1}$$

The input is expressed as the difference in step function.

$$Q(t) = 90[u(t) - u(t-0.1)]$$

The transform of this is

$$Q(s) = \frac{90}{s}(1 - e^{-0.1s})$$

Combing this and the transfer function of the ~~response~~ process,

$$H(s) = 10\left(\frac{1}{s(s+1)} - \frac{e^{-0.1s}}{s(s+1)}\right) \text{ ——— (1)}$$

Inversion of 1st term is $10(1-e^{-t})$

Inversion of 2nd term is by theorem on translation of functions

Inverse of $e^{-st_0} f(s)$ is $f(t-t_0)$ with $f(t)=0$ for $t-t_0<0$ or $t<t_0$. The inverse of the 2nd term in eq. ② is,

$$\therefore L^{-1}\left\{\frac{e^{-0.1s}}{s(s+1)}\right\} = 0 \quad \text{for } t<0$$

$$= 10\left[1-e^{-(t-0.1)}\right] \quad \text{for } t>0.1$$

The complete solution to this problem, which is the inverse of eq. ① is,

$$H(t) = 10(1-e^{-t}) \qquad t<0.1$$

$$H(t) = 10\{(1-e^{-t})-[1-e^{-(t-0.1)}]\} \qquad t>0.1$$

Simplifying the expression for $H(t)$ for $t>0.1$ gives

$$H(t) = 1.052\,e^{-t} \qquad t>0.1$$

The response of the system to an impulse of magnitude 9 is given by,

$$H(t)\Big|_{\text{impulse}} = 9\left(\frac{1}{9}\right)e^{-t} = e^{-t}$$

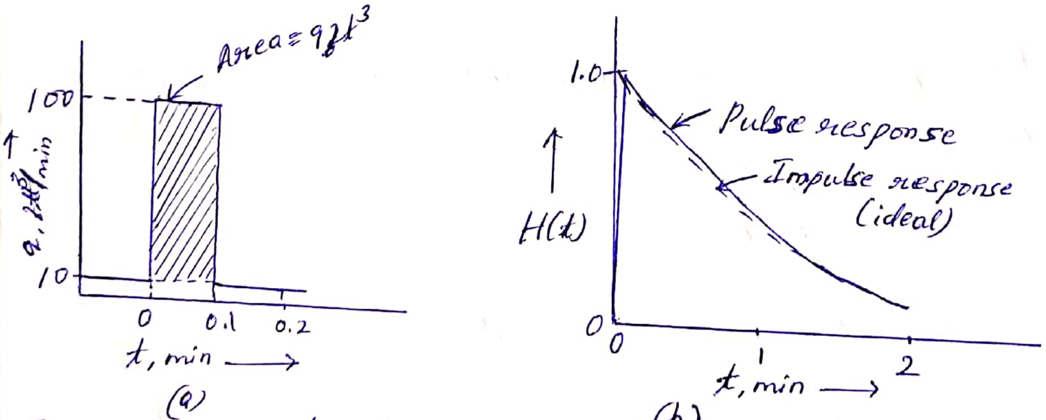

Fig - Approximation of an impulse function in a liquid-level system.
a) Pulse input, b) Response of tank level.

<u>Note</u> - The responses to step and sinusoidal forcing functions are the same for the liquid-level system as for the mercury thermometer. This is the advantage of characterizing all first-order systems by the same transfer function.

Liquid-Level Process with Constant-flow Outlet

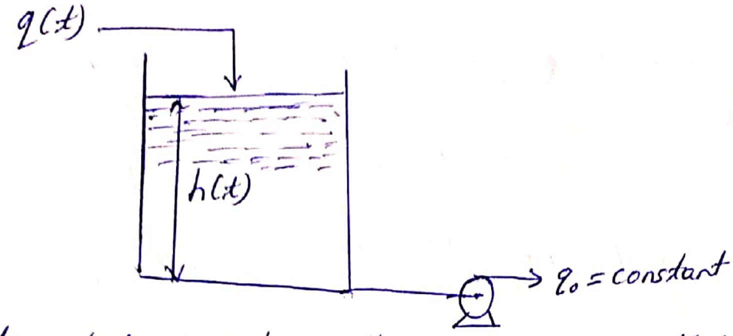

Fig- Liquid-level system with constant flow outlet

An example of a transfer function that often arises in control systems may be developed by considering the liquid-level system shown in fig. The resistance is replaced by a constant-flow pump. Assume that cross-sectional area and density are constant.

WKT, mass balance is,

$$q(t) - q_o(t) = A \frac{dh}{dt} \quad \text{——} \quad ①$$

$q(t)$ — is now constant, $\therefore q(t) - q_o = A\frac{dh}{dt}$ —— ②

At steady state eqn ② becomes,

$$q_{(s)} - q_o = 0 \quad \text{——} \quad ③$$

Subtracting eq. ② from eq③ and introducing the deviation variables,

$$Q = q - q_s$$
$$H = h - h_s$$

$$Q = A \cdot \frac{dH}{dt} \quad \text{——} \quad ④$$

Taking the Laplace transform of each side of eq④ and solving for H/Q gives,

$$Q(s) = A.s.H(s) - H(0)$$

$$\therefore \frac{H(s)}{Q(s)} = \frac{1}{As} \quad \text{——} \quad ⑤$$

The transfer function, $\frac{1}{As}$ in eq. ⑤ is equivalent to integration.

Therefore solution is,

$$h(t) = h_s + \frac{1}{A}\int_0^t Q(t)\,dt \quad \text{——} \quad ⑥$$

If a step change $Q(t) = u(t)$ is applied to the system, the result is,

$$h(t) = h_s + \frac{t}{A} \quad \text{——} \quad ⑦$$

The step response given by eqn ⑦ is a ramp function that grows without limit. Such a system that grows without limit for a sustained change in input is said to have nonregulation.

Systems that have a limited change in output for a sustained change in input are said to have "regulation".

Ex - step response of a first-order system.

The transfer function for the liquid-level system with constant outlet flow is,

$$\frac{H(s)}{Q(s)} = \frac{1}{As}$$

This can be considered as a special case of

$$\frac{H(s)}{Q(s)} = \frac{R}{\tau s+1}$$

as $R \longrightarrow \infty$

$$\frac{H(s)}{Q(s)} = \frac{1}{As}$$

Mixing Process

Consider the mixing process as shown in fig. in which a stream of solution containing dissolved salt flows at a constant volumetric flow rate q into a tank of constant holdup volume V. The concentration of the salt in the entering stream, x (mass of salt/volume), varies with time.

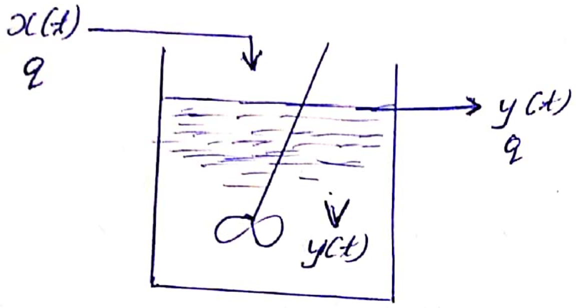

Fig - Mixing process

It is desired to determine the transfer function relating the outlet concentration y to the inlet concentration x.

Assuming the density of the solution to be constant, the flow rate in must equal the flow rate out, since the holdup volume is fixed.

Transient mass balance for the salt,

Flow rate of salt in − Flow rate of salt out = Rate of accumulation of salt in the tank.

$$qx - qy = \frac{d(Vy)}{dt} \quad \text{—— (1)}$$

Introducing the deviation variables at s.s

$$qx_s - qy_s = 0 \quad \text{—— (2)}$$

Subtracting eq. (2) from (1) and introducing the deviation variables,

$$X = x - x_s$$
$$Y = y - y_s$$

give,

$$qX - qY = V.\frac{dY}{dt} \quad \text{--- (3)}$$

Taking the Laplace transform.

$$q\,X(s) - q\,Y(s) = V[s\,Y(s) - Y(0)]$$

$$qX(s) = Y(s)[q + Vs]$$

$$\frac{Y(s)}{X(s)} = \frac{q}{q+Vs} = \frac{1}{1+\frac{V}{q}s}$$

$$\frac{Y(s)}{X(s)} = \frac{1}{\tau s+1} \quad \text{--- (4)}$$

where, $\tau = \frac{V}{q}$

Summary:

In each example, of a first-order system, the time constant has been expressed in terms of system parameters, thus

For thermometer, $\tau = \frac{mc}{hA}$

For liquid-level process, $\tau = AR$

For mixing process, $\tau = \frac{V}{q}$

LINEARIZATION

Characterization of a dynamic system by a transfer function can be done only for linear systems. (those described by linear differential equations).

Consider liquid-level system and approximated for linearization. Assume that the resistance follows the square-root relationship.

$$q_o = C h^{1/2} \quad \text{——} \quad (1)$$

where, C is constant.

For a liquid of constant density and a tank of uniform c/s area A, material balance around the tank gives,

$$q(t) - q_o(t) = A\frac{dh}{dt} \quad \text{——} \quad (2)$$

Combinig eq. (1) and (2) gives the non-linear differential eq.

$$q - Ch^{1/2} = A\frac{dh}{dt} \quad \text{——} \quad (3)$$

As non-linear term $h^{1/2}$ is present, Laplace transform cann't be applied.

By means of a Taylor-series expansion, the function $q_o(h)$ can be expanded around the steady-state value h_s, thus,

$$q_o = q_o(h_s) + q_o'(h_s)(h-h_s) + \frac{q_o''(h_s)(h-h_s)^2}{2!} + \text{------} \quad \text{——} \quad (4)$$

where,

$q_o'(h_s)$ – is 1st derivative of q_o evaluated at h_s.

$q_o''(h_s)$ – is 2nd derivative of q_o evaluated at h_s -- etc

Considering only linear term, the result is,

$$q_o \cong q_o(h_s) + q_o'(h_s)(h-h_s) \quad \text{——} \quad (5)$$

Taking the derivative of q_o with respect to h in eq. (1) and evaluating the derivative at $h=h_s$ gives,

$$q_0'(h_s) = \frac{1}{2} C h_s^{-1/2} \quad \text{——} \ (6)$$

Introducing eq. (6) into eq. (5) gives

$$q_0 = q_{0s} + \frac{1}{R_1}(h - h_s) \quad \text{——} \ (7)$$

where, $q_{0s} = q_0(h_s)$

$$(R_1)^{-1} = \frac{1}{2} C h_s^{-1/2}$$

Substituting eq. (7) into eq. (2) gives

$$q - q_{0s} - \frac{h - h_s}{R_1} = A\frac{dh}{dt} \quad \text{——} \ (8)$$

At steady state the flow entering the tank equals the flow leaving the tank; ~~th~~us,

$$q_0 = q_{0s} \quad \text{——} \ (9)$$

Introducing eq. (9) into eq. (8) gives,

$$A\frac{dh}{dt} + \frac{h - h_s}{R_1} = q - q_s \quad \text{——} \ (10)$$

Introducing the deviation variables

$$\left.\begin{aligned} Q &= q - q_s \\ H &= h - h_s \end{aligned}\right\}$$ into eqn (10) and on transforming gives,

$$\frac{H(s)}{Q(s)} = \frac{R_1}{\tau s + 1} \quad \text{——} \ (11)$$

where, $R_1 = 2h_s^{1/2}/C$

$\tau = R_1 A$

The transfer function obtained is identical in form with that of the linear system. i.e $\frac{H(s)}{Q(s)} = \frac{R}{\tau s + 1}$.

However, in this case, the resistance R_1 depends on the steady state conditions around which the process operates. Graphically, the resistance R_1 is the reciprocal of the slope of the tangent line passing through the point

$(q_{os}\ h_s)$ as in fig.

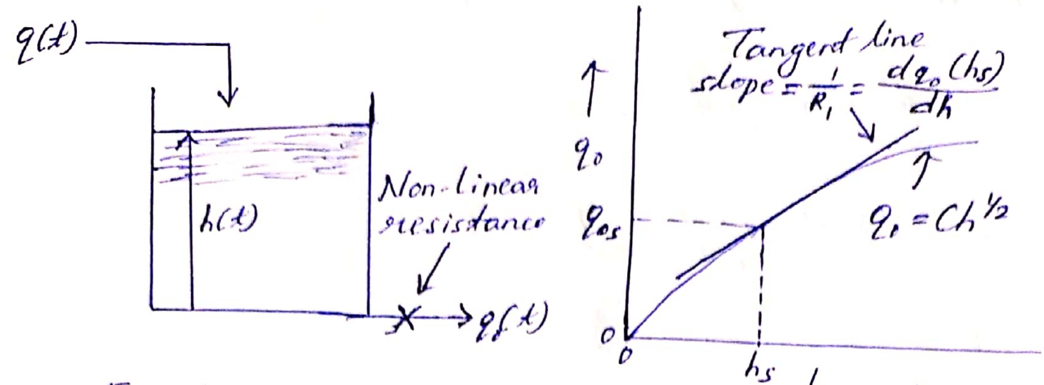

Fig- Liquid-level system with non-linear resistance

Furthermore, the linear approximation given by eq.⑤ is the equation of ~~change~~ the tangent line itself. From the graphical representation, it should be clear that the linear approximation improves as the deviation in h becomes smaller.

If an analytic expression such as $h^{1/2}$ for the nonlinear function, but only a graph of the function, the technique can still be applied by representing the function by the tangent line passing through the point of operation.

Wheather or not the linearized result is a valid representation depends on the operation of the system. If the level is being maintained by a controller at or close to a fixed level h_s, then by the very nature of the control imposed on the system, deviations in level should be small (for good control) and the linearized eq. is adequate.

On the other hand, if the level should change over a wide range, linear approximation may be very poor and the system may deviate significantly from the prediction of the linear transfer function. In such cases, it may be necessary to use the more difficult methods of non-linear analysis.

In summary, we have characterized, in an approximate sense, a non-linear system by a linear transfer function. In general this technique may be applied to any nonlinearity that can be expressed in a Taylor series (or equivalently, has a unique slope at the operating point). Since this includes most nonlinearities arising in process control.

RESPONSE OF FIRST-ORDER SYSTEMS IN SERIES

First-order systems in series

Non-Interacting System

Consider the liquid-level system as shown in fig. in which two tanks are arranged so that the outlet flow from the first tank is the inlet flow to the second tank.

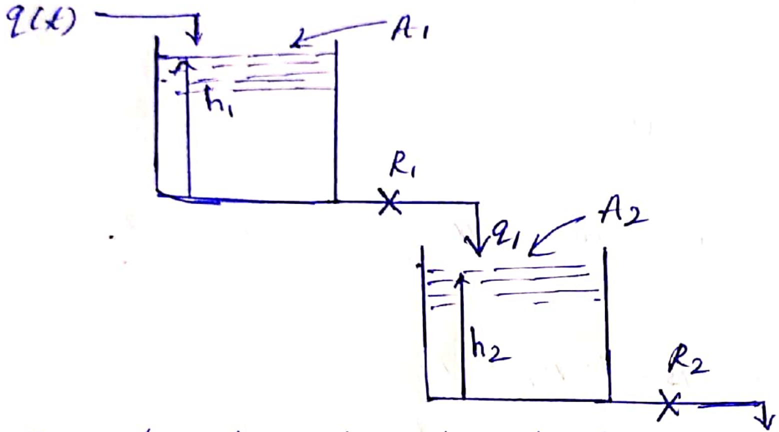

Fig- Noninteracting two-tank liquid-level systems

The outlet flow from tank 1 discharges directly into the atmosphere before spilling into tank 2 and the flow through R_1 depends only on h_1. The variation in h_2 in tank 2 does not affect the transient response occuring in tank 1. This type of system is referred to as a non-interacting system.

Assume that liquid has constant density, tanks have uniform c/s area, and the flow resistances ~~to be~~ are linear.

Transfer functions for each tank, $Q_1(s)/Q(s)$ and $H_2(s)/Q_1(s)$ is obtained by transient mass balance around each tank; then these transfer functions will then be combined to eliminate the intermediate flow $Q_1(s)$ and produce the desired transfer function.

Balance on Tank-1 gives,

$$q - q_1 = A_1 \frac{dh_1}{dt} \quad \text{——} ①$$

Balance on Tank-2 gives,

$$q_1 - q_2 = A_2 \frac{dh_2}{dt} \quad \text{——} ②$$

The flow head relationships for the two linear resistances are given by.

$$q_1 = \frac{h_1}{R_1} \quad \text{——} ③$$

$$q_2 = \frac{h_2}{R_2} \quad \text{——} ④$$

Combining eqs. ① and ③ and introducing the deviation variables,

$$(q - q_s) = \frac{1}{R_1}(h_1 - h_{1s}) + A_1 \frac{d(h_1 - h_{1s})}{dt} \quad \text{——} ⑤$$

$$Q = \frac{1}{R_1} H_1 + A_1 \frac{dH_1}{dt}$$

$$Q(s) = \frac{H_1(s)}{R_1} + A_1 s . H_1(s)$$

$$Q(s) = H_1(s)\left[\frac{1}{R_1} + A_1 s\right]$$

$$\frac{H_1(s)}{Q(s)} = \frac{R_1}{A_1 R_1 s + 1} = \frac{R_1}{\tau_1 s + 1} \quad \text{——} ⑥$$

where, $\tau_1 = A_1 R_1$

If, $Q(t) = u(t) \Rightarrow Q(s) = \frac{1}{s}$

$$H_1(s) = \frac{1}{s} \frac{R_1}{\tau_1 s + 1}$$

$$H(t)\Big|_{t=\infty} = \lim_{s\to 0}[s H(s)] = \lim_{s\to 0} \frac{R_1}{\tau_1 s + 1} = R_1$$

$$q_{1s} = \frac{h_{1s}}{R_1}, \quad Q_1 = q_1 - q_{1s}$$

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$$Q_1 = \frac{H_1}{R_1}$$

$$Q_1(s) = \frac{H_1(s)}{R_1} \quad \text{—— (7)}$$

Eqn (7) in (6)

$$\frac{Q_1(s) R_1}{Q(s)} = \frac{R_1}{T_1 s + 1}$$

$$\frac{Q_1(s)}{Q(s)} = \frac{1}{T_1 s + 1} \quad \text{—— (8)}$$

Similarlly, combining eqn. (2) and (4), the transfer function for tank 2 is obtained as,

$$\frac{H_2(s)}{Q_1(s)} = \frac{R_2}{T_2 s + 1} \quad \text{—— (9)}$$

where, $H_2 = h_2 - h_{2s}$

$T_2 = R_2 A_2$

The overall transfer function $H_2(s)/Q(s)$ is obtained by multiplying eqs. (8) and (9) to eliminate $Q_1(s)$;

$$\frac{H_2(s)}{Q(s)} = \frac{1}{T_1 s + 1} \cdot \frac{R_2}{T_2 s + 1} \quad \text{—— (10)}$$

Eq. (10) is the overall transfer function and it is product of two first-order transfer functions, each one of which is the transfer function of a single tank operating independently of the other.

Example -

Two noninteracting tanks are connected in series as in fig. The time constants are $\tau_2 = 1$ and $\tau_1 = 0.5$; $R_2 = 1$. Sketch the response of the level in tank 2 if a unit-step change is made in the inlet flow rate to tank 1.

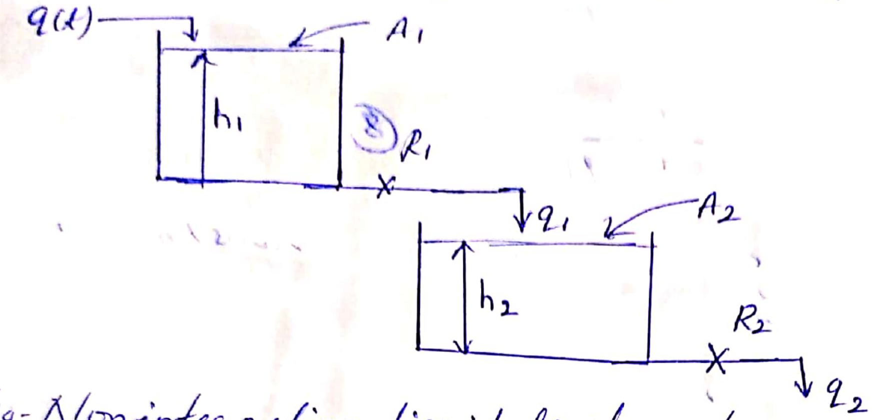


Fig - Noninteracting liquid-level system.

Solution:

The transfer function for noninteracting system is

$$\frac{H_2(s)}{Q(s)} = \frac{R_2}{(\tau_1 s+1)(\tau_2 s+1)}$$

For a unit step change in Q,

$$H_2(s) = \frac{1}{s}\cdot\frac{R_2}{(\tau_1 s+1)(\tau_2 s+1)}$$

Inversion by partial-fraction expansion.

$$H_2(t) = R_2\left[1 - \frac{\tau_1\tau_2}{\tau_1-\tau_2}\left(\frac{1}{\tau_2}e^{-t/\tau_1} - \frac{1}{\tau_1}e^{-t/\tau_2}\right)\right]$$

Substituting the values of τ_1, τ_2 and R_2

i.e $\tau_1 = 0.5$, $\tau_2 = 1$, $R_2 = 1$

$$\therefore H_2(t) = 1 - (2e^{-t} - e^{-2t})$$

The response is S-shaped and the slope $\frac{dH_2}{dt}$ at origin is zero. If the change in flow rate were introduced into the second tank, the response would be 1st order and is shown for comparison in fig. by dotted curve.

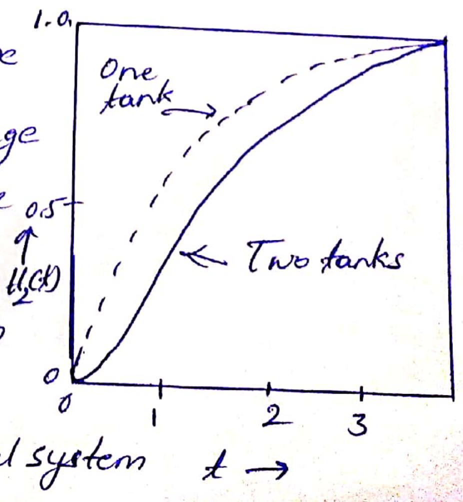


Fig - Transient response of liquid-level system

Generalization for several Noninteracting Systems in Series

The overall transfer function for two noninteracting first-order systems connected in series is simply the product of the individual transfer functions.

For n-noninteracting 1st order systems represented by block diagram.

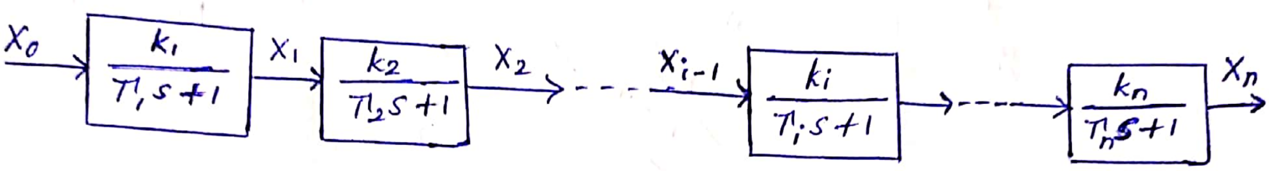


Fig – Noninteracting first-order systems.

The block diagram is equivalent to the relationships

$$\frac{X_1(s)}{X_0(s)} = \frac{k_1}{\tau_1 s+1}$$

$$\frac{X_2(s)}{X_1(s)} = \frac{k_2}{\tau_2 s+1}$$

- - - - - - - - - - - - - -

$$\frac{X_n(s)}{X_{n-1}(s)} = \frac{k_n}{\tau_n s+1}$$

To obtain the overall transfer function, multiply together the individual transfer functions; thus

$$\frac{X_n(s)}{X_0(s)} = \prod_{i=1}^{n} \frac{k_i}{\tau_i s+1}$$

W.K.T, the step response of a system consisting of two first-order systems is S-shaped and that the response changes very slowly just after introduction of the step input. This sluggishness (OR) delay is sometimes called transfer lag. and is always present when two (OR) more first-order systems are connected in series.

For a single first-order systems there is no transfer lag; i.e the response begins immediately after the step change is applied and the rate of change of response (slope of response curve is maximal at $t=0$.

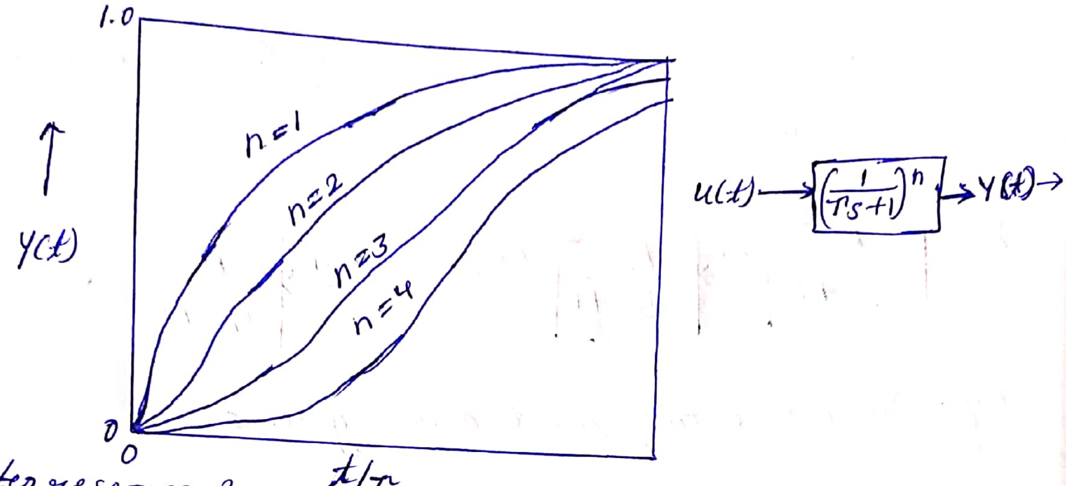


Fig- Step response of noninteracting first-order systems.

In order to show how the transfer lag is increased as the number of stages increases. Fig. gives the unit-step response curves for several systems containing one or more first-order stages in series.

Interacting System:

Consider the liquid-level systems as shown in fig. in which two tanks are arranged so that the outlet flow from the first tank is the inlet flow to the second tank.

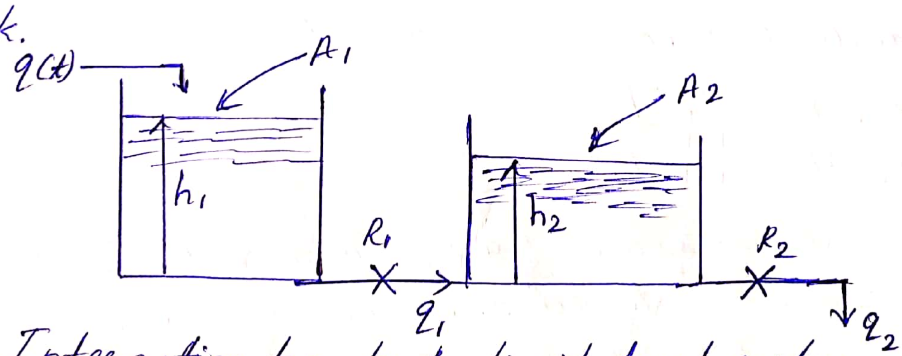


Fig- Interacting two-tank liquid-level system.

The outlet flow from tank 1 discharges directly into the tank 2 and the flow through R_1 depends on the difference between h_1 and h_2.

Assume that liquid has constant density, tanks have uniform c/s area, and the flow resistances are linear.
Transfer functions for each tank $Q_1(s)/Q(s)$ and $H_2(s)/Q_1(s)$ is obtained by transient mass balance around each tank; then these transfer functions will then be combined to eliminate the intermediate flow $Q_1(s)$ and produce the desired transfer function.

mass balance on tank 1 and 2 respectively,

$$q - q_1 = A_1 \frac{dh_1}{dt} \quad \text{—①}$$

$$q_1 - q_2 = A_2 \frac{dh_2}{dt} \quad \text{—②}$$

The flow-head relationship for tank 1 is now,

$$q_1 = \frac{1}{R_1}(h_1 - h_2) \quad \text{—③}$$

$$q_2 = \frac{h_2}{R_2} \quad \text{—④}$$

A simple way to combine eqs. ①, ②, ③ and ④ is to first express them in terms of deviation variables, transform the resulting eqns, and then combine the transformed eqns. to eliminate the unwanted variables.

At steady state eqn ① and ②

$$q_s - q_{1s} = 0 \quad \text{—⑤}$$

$$q_{1s} - q_{2s} = 0 \quad \text{—⑥}$$

Subtracting eq. ⑤ from eq. ① & eq. ⑥ from eq. ② and introducing the deviation variables give.

$$(q - q_s) - (q_1 - q_{1s}) = A_1 \frac{d(h_1 - h_{1s})}{dt}$$

$$Q - Q_1 = A_1 \frac{dH_1}{dt} \quad \text{—⑦}$$

llly

$$(q_1 - q_{1s}) - (q_2 - q_{2s}) = A_2 \frac{d(h_2 - h_{2s})}{dt}$$

$$Q_1 - Q_2 = A_2 \frac{dH_2}{dt} \quad \text{—⑧}$$

Expressing eqs. ③ and ④ in terms of deviation variables,

$$(q_1 - q_{1s}) = \frac{1}{R_1}\left[(h_1 - h_{1s}) - (h_2 - h_{2s})\right]$$

$$Q_1 = \frac{H_1 - H_2}{R_1} \quad \text{——— (9)}$$

$$(q_2 - q_{2s}) = \frac{h_2 - h_{2s}}{R_2}$$

$$Q_2 = \frac{H_2}{R_2} \quad \text{——— (10)}$$

Transforming eqn. ⑦, ⑧, ⑨ and ⑩ gives,

⑦ $\Rightarrow$ $Q(s) - Q_1(s) = A_1 s H_1(s)$ ——— (11)

⑧ $\rightarrow$ $Q_1(s) - Q_2(s) = A_2 s H_2(s)$ ——— (12)

⑨ $\Rightarrow$ $R_1 Q_1(s) = H_1(s) - H_2(s)$ ——— (13)

⑩ $\Rightarrow$ $R_2 Q_2(s) = H_2(s)$ ——— (14)

The analysis has produced four algebraic eqs. containing five unknowns. (Q, Q_1, Q_2, H_1 and H_2). These eqs. are combined to eliminate Q_1, Q_2, and H_1 and arrive at the desired transfer function:

$$\frac{H_2(s)}{Q(s)} = \frac{R_2}{\tau_1 \tau_2 s^2 + (\tau_1 + \tau_2 + A_1 R_2)s + 1} \quad \text{——— (15)}$$

The term $A_1 R_2$ is present in the coefficient of s for interacting system but its not present in noninteracting system.

The term interacting is often referred to as "loading". The second tank is said to load first tank.

The effect of interaction on the transient response of a system, Consider a two-tank system for which the time constants are equal ($\tau_1 = \tau_2 = \tau$).

If the tanks are noninteracting, the transfer function relating inlet flow to outlet flow is,

$$\frac{Q_2(s)}{Q(s)} = \left(\frac{1}{\tau s+1}\right)^2 \quad \text{—— (16)}$$

The unit step response for this transfer function can be obtained by the usual procedure to give,

$$Q_2(t) = 1 - e^{-t/\tau} - \frac{1}{\tau} e^{-t/\tau} \quad \text{—— (17)}$$

If the tanks are interacting, the overall transfer function according to eq. (15) is (assuming $A_1 = A_2$)

$$\frac{Q_2(s)}{Q(s)} = \frac{1}{\tau^2 s^2 + 3\tau s + 1} \quad \text{—— (18)}$$

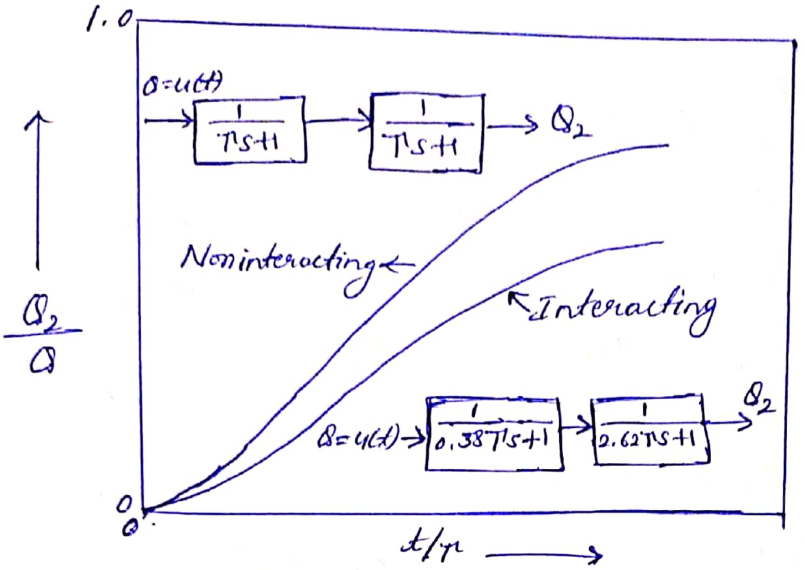
By application of the quadratic formula, the denominator of this transfer function is,

$$\frac{Q_2(s)}{Q(s)} = \frac{1}{(0.38\tau s+1)(2.62\tau s+1)} \quad \text{—— (19)}$$

The effect of interaction has been to change the effective time constants of the interacting system. One time constant has become considerably larger and the other smaller than the time constant τ of either tank in the noninteracting system. The response of $Q_2(t)$ to a unit-step change in $Q(t)$ for the interacting case is,

$$Q_2(t) = 1 + 0.17 e^{-t/0.38\tau} - 1.17 e^{-t/2.62\tau} \quad \text{—— (20)}$$

The unit step responses (eq. (17) and (20)) for the noninteracting and interacting systems are plotted to show the effect of interaction.
From this fig. observed that interaction slows up the response.



This result understood on physical grounds in following way:
If the same size step change is introduced into the two systems of interacting / noninteracting systems, the flow from tank 1 (q_1) for the noninteracting case will not be reduced by the increase in level in tank 2.

However, for the interacting case, the flow q_1 will be reduced by the build-up of level in tank 2. At any time t, following the introduction of the step input, q_1 for the interacting case will be less than for the noninteracting case with the result that h_2 (or q_2) will increase at a slower rate.

In general, the effect of interaction on a system containing two-first-order lags is to change the ratio of effective time constants in the interacting system. In terms of the transient response, this means that the interacting system is more sluggish than the noninteracting system.

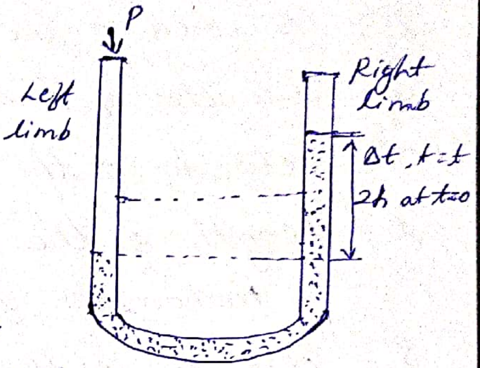
HIGHER-ORDER SYSTEMS: SECOND-ORDER AND TRANSPORTATION LAG

SECOND-ORDER SYSTEM:

If the dynamics of the system are represented by second order differential equation then it is known as a second order control system.

Transfer function for manometer

Consider a manometer as shown in fig. Density of manometer fluid is ρ, the pressure P is acting in the left limb of manometer tube.



At time $t=0$, the manometer is at steady state. The pressure P is applied to the left limb and 'h' is the height of the manometer fluid above the rest position.

Assuming that the gas density is very small than the density of the liquid and the flow of the fluid in the manometer is laminar under these conditions the various forces are acting on the manometer fluid.

The forces in the direction of the motion of the manometer are taken as positive and the others opposing the direction of the motion of the manometer fluid are taken as negative.

The following forces are acting on the manometer liquid

Force due to pressure P in the left limb $= PA$

Force due to manometer liquid in the right limb under the influence of gravity $= -2h\rho A g$

$m \cdot \frac{kg}{m^3} \cdot m^2 \cdot \frac{m}{s^2} = kg\frac{m}{s^2}$

Force due to friction b/w the manomer tube and the liquid in the manometer due to the motion of liquid $= -\Delta H_F \rho g A$

$m \cdot \frac{kg}{m^3} \cdot \frac{m}{s^2} \cdot m^2 = \frac{kg\, m}{s^2}$

Since ΔH_F head lost due to friction $= 4f\left(\frac{l}{d}\right)\left(\frac{u^2}{2g}\right)$

and the friction $f = \frac{16}{N_{Re}}$. for the laminar flow where

$N_{Re} = \frac{du\rho}{\mu}$ or $f = \frac{16\mu}{du\rho}$

$\frac{m}{m} \times \frac{m^2}{s^2} \cdot \frac{s^2}{N} \frac{kg}{m^3} \times \frac{m}{s^2} \cdot m^2$
$= kg \cdot \frac{m}{s^2}$

$\therefore$ Force due to friction $= -4\left(\frac{16\mu}{du\rho}\right)\left(\frac{l}{d}\right)\left(\frac{u^2}{2g}\right) \cdot \rho g \cdot A$

On solving force due to friction $= \left(\frac{-32\mu l A}{d^2}\right)\frac{dh}{dt}$

The variables are as follows:

A – c/s area of the manometer tube, m^2

d – diameter of the manometer ~~fluid, m/s~~ m^2

u – velocity of the manometer fluid, m/s

l – length of the manometer tube containing manometer fluid, m

ρ – density of the manometer fluid, kg/m^3

μ – viscosity of the manometer fluid, $N\text{-}s/m^2$

Rate of change of momentum $= lA \cdot \rho \frac{d^2h}{dt^2}$

where, the mass of the fluid in the manometer, $= l \cdot A \cdot \rho$

and acceleration of the fluid $= \frac{d^2h}{dt^2}$.

Newton's law of motion states that the sum of all forces acting is equal to the rate of change of momentum.

Net force acting = Rate of change of momentum.

$$PA - 2h\rho A g - \frac{32\mu A L}{d^2} \cdot \frac{dh}{dt} = l \cdot A \cdot \rho \cdot \frac{d^2h}{dt^2}$$

$\div\ 2A\rho g$ on both sides

$$\frac{P}{2\rho g} - h - \frac{16\mu l}{d^2 \rho g} \cdot \frac{dh}{dt} = \frac{l}{2g} \cdot \frac{d^2h}{dt^2}$$

$$\tau^2 = \frac{l}{2g}, \quad 2\phi\tau = \frac{16\mu l}{d^2\rho g}$$

where, τ – is the time constant & ϕ – is damping parameter.

$$\tau^2 \frac{d^2h}{dt^2} + 2\phi\tau \frac{dh}{dt} + h = X(t)$$

This eqn. represents the dynamic behaviour of the manometer which is second order differential eqn.

Therefore, manometer is a second order system.

where, the input variable,

$$X(t) = \frac{P}{2\rho g}$$

$$\tau^2 s^2 H(s) + 2\phi\tau s H(s) + H(s) = X(s)$$

(OR) $$H(s)\left[\tau^2 s^2 + 2\phi\tau s + 1\right] = X(s)$$

T. F of the manometer is given as,

$$G(s) = \frac{H(s)}{X(s)} = \frac{1}{\tau^2 s^2 + 2\phi\tau s + 1}$$

where, $\tau = \sqrt{\frac{l}{2g}}$, $\phi = \frac{8\mu L}{d^2 \rho g}\sqrt{\frac{2g}{l}}$ ϕ (OR) ξ.

τ - time constant for the manometer has unit of time

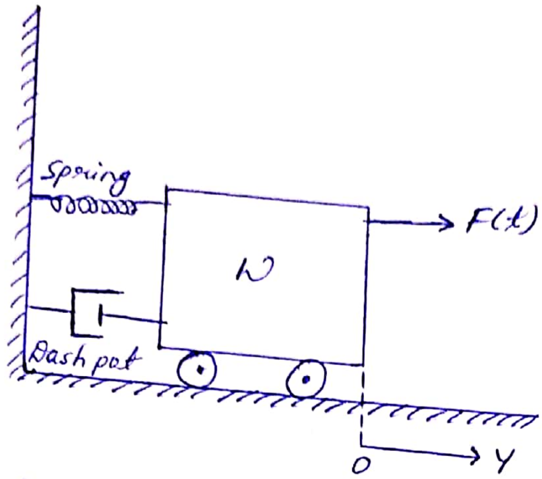
ξ (OR) ϕ - is damping co-efficient of manometer and it is dimensionless.

It represents the character of the second order control system.

SECOND ORDER SYSTEM OR QUADRATIC LAG

TRANSFER FUNCTION.

Ex – Damped Vibrator



A block of mass W resting on a horizontal, frictionless table is attached to a linear spring. A viscous damper (dashpot) is also attached to the block.
Assume that the system is free to oscillate horizontally under the influence of a forcing function F(t). The origin of the co-ordinate system is taken as the right edge of the block is assumed to be at rest at this origin, when the spring is in the relaxed ⊗ unstretched condition.

Assume block is at rest at origin at $t=0$.
Arrows indicate the directions for force and displacement.
Consider the block at some instant when it is to the right of $Y=0$ and when it is moving toward the right (Positive direction). under these conditions, the position Y and velocity $\frac{dY}{dt}$ are both positive. At this particular instant, the following forces are acting on the block:

1. The force exerted by the spring (towards left) of $-KY$ where K is a the constant, called Hooke's constant.
2. The viscous friction force (acting to the left) of $-C\frac{dY}{dt}$, where C is a positive constant called the damping co-efficient.

3. The external force F(t) (acting toward the right).

Newton's law of motion – The sum of all forces acting on the mass is equal to the rate of change of momentum (m x a)

$$W\frac{d^2Y}{dt^2} = -KY - C\frac{dY}{dt} + F(t) \quad \text{—} \ (1)$$

$$W\frac{d^2Y}{dt^2} + C\frac{dY}{dt} + KY = F(t) \quad \text{—} \ (2)$$

where,

W – mass of block

C – viscous damping co-efficient.

K – Hooke's constant

F(t) – Driving force, a function of time

Eqn (2) ÷ K

$$\frac{W}{K}\frac{d^2Y}{dt^2} + \frac{C}{K}\frac{dY}{dt} + Y = \frac{F(t)}{K} \quad \text{—} \ (3)$$

For convenience,

$$\tau^2\frac{d^2Y}{dt^2} + 2\zeta\tau\frac{dY}{dt} + Y = X(t) \quad \text{—} \ (4)$$

where, $\tau^2 = \frac{W}{K}$ — (5), $2\zeta\tau = \frac{C}{K}$ — (6), $X(t) = \frac{F(t)}{K}$ — (7)

$\tau = \sqrt{\frac{W}{K}}$ sec — (8) $\zeta = \sqrt{\frac{C^2}{4WK}}$ — (9) dimensionless

τ & ζ are +ve (By defination).

If the block is motionless ($dY/dt = 0$) and located at its rest position ($Y=0$) before the forcing function is applied, then L.T of eqn.

$$\tau^2 s^2 Y(s) + 2\zeta\tau s Y(s) + Y(s) = X(s) \quad \text{—} \ (10)$$

$$\frac{Y(s)}{X(s)} = \frac{1}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

Two parameters τ and ζ are required to characterize the dynamics of a second-order system.

Step Response;

If the forcing function is a unit-step function,

$$X(s) = \frac{1}{s} \quad \text{——} \; (1)$$

For a damped vibrator, this is equivalent to suddenly applying a force of magnitude K directed toward the right at time $t=0$. This follows from the fact that X is defined by the relationship $X(t) = \frac{F(t)}{K}$.

Super-position will enable us to determine easily the response to a step function of any other magnitude.

WKT
$$\frac{Y(s)}{X(s)} = \frac{1}{\tau^2 s^2 + 2\zeta\tau s + 1} \quad \text{——} \; (2)$$

Combing eq. (1) with the T.F (2) gives

$$Y(s) = \frac{1}{s} \cdot \frac{1}{\tau^2 s^2 + 2\zeta\tau s + 1} \quad \text{——} \; (3)$$

The quadratic term in this eqn. may be factored into two linear terms that contain the roots.

$$\frac{-2\zeta\tau \pm \sqrt{4\zeta^2\tau^2 - 4\tau^2 \cdot 1}}{2\tau^2}$$

$$s_1 = -\frac{\zeta}{\tau} + \frac{\sqrt{\zeta^2 - 1}}{\tau} \quad \text{——} \; (4) \qquad -\frac{\zeta}{\tau} \pm \frac{2\tau\sqrt{\zeta^2-1}}{2\tau^2}$$

$$s_2 = -\frac{\zeta}{\tau} - \frac{\sqrt{\zeta^2 - 1}}{\tau} \quad \text{——} \; (5) \qquad -\frac{\zeta}{\tau} \pm \frac{\sqrt{\zeta^2-1}}{\tau}$$

$$Y(s) = \frac{1/\tau^2}{(s)(s-s_1)(s-s_2)} \quad \text{——} \; (6)$$

The response of the system $Y(t)$ can be found by inverting eqn. (6). The roots s_1 and s_2 will be real OR complex depending on the parameter ζ. The nature of the roots will in turn, affect the form of $Y(t)$.

The problem may be divided into 3 cases.

Case 1 - step response for $\zeta > 1$

Case 2 - Step response for $\zeta < 1$

Case 3 - Step response for $\zeta = 1$

CASE I – STEP RESPONSE FOR $\zeta > 1$

For this case,

$$Y(s) = \frac{1/\tau^2}{(s)(s-s_1)(s-s_2)}$$

$$y(t) = \frac{1}{\tau^2} L^{-1}\left[\frac{C_1}{s} + \frac{C_2}{s-s_1} + \frac{C_3}{s-s_2}\right]$$

$$\frac{1}{(s)(s-s_1)(s-s_2)} = \frac{C_1}{s} + \frac{C_2}{s+\frac{\zeta}{\tau} - \frac{\sqrt{\zeta^2-1}}{\tau}} + \frac{C_3}{s+\frac{\zeta}{\tau}+\frac{\sqrt{\zeta^2-1}}{\tau}}$$

$$1 = C_1(s-s_1)(s-s_2) + C_2(s)(s-s_2) + C_3(s)(s-s_1)$$

Put $s-s_1=0 \Rightarrow s=s_1$

$1 = C_2(s_1)(s_1-s_2)$, $\quad C_2 = \dfrac{1}{s_1(s_1-s_2)}$

$s=0$,

$1 = C_1(-s_1)(-s_2)$

$C_1 = \dfrac{1}{s_1 s_2}$

$s-s_2=0$

$s=s_2$

$1 = C_3(s_2)(s_2-s_1)$

$C_3 = \dfrac{1}{s_2(s_2-s_1)}$

$$y(t) = \frac{1}{\tau^2} L^{-1}\left[\frac{1}{s_1 s_2 s} + \frac{1}{s_1(s_1-s_2)(s-s_1)} + \frac{1}{s_2(s_2-s_1)(s-s_2)}\right]$$

$$= \frac{1}{\tau^2}\left[\frac{1}{s_1 s_2} + \frac{e^{s_1 t}}{s_1(s_1-s_2)} + \frac{e^{s_2 t}}{s_2(s_2-s_1)}\right]$$

$$= \frac{1}{\tau^2}\left[\tau^2 + \frac{\tau^2 e^{s_1 t}}{(2\sqrt{\zeta^2-1})(-\zeta+\sqrt{\zeta^2-1})} + \frac{\tau^2 e^{s_2 t}}{(\zeta+\sqrt{\zeta^2-1})(2\sqrt{\zeta^2-1})}\right]$$

$$y(t) = 1 + \frac{e^{t\left(\frac{-\zeta+\sqrt{\zeta^2-1}}{\tau}\right)}}{(2\sqrt{\zeta^2-1})(-\zeta+\sqrt{\zeta^2-1})} + \frac{e^{t\left(\frac{-\zeta-\sqrt{\zeta^2-1}}{\tau}\right)}}{(\zeta+\sqrt{\zeta^2-1})(2\sqrt{\zeta^2-1})}$$

$$= 1 + \frac{e^{-\zeta\frac{t}{\tau}} \cdot e^{\frac{t\sqrt{\zeta^2-1}}{\tau}}}{2\sqrt{\zeta^2-1}(-\zeta+\sqrt{\zeta^2-1})} + \frac{e^{-\zeta\frac{t}{\tau}} \cdot e^{-\frac{\sqrt{\zeta^2-1}}{\tau}\cdot t}}{2\sqrt{\zeta^2-1}(\zeta+\sqrt{\zeta^2-1})}$$

$$= 1 + \frac{e^{-\zeta\frac{t}{\tau}}}{2\sqrt{\zeta^2-1}}\left[\frac{e^{\frac{\sqrt{\zeta^2-1}}{\tau}\cdot t}}{(-\zeta+\sqrt{\zeta^2-1})} + \frac{e^{-\frac{\sqrt{\zeta^2-1}}{\tau}\cdot t}}{(\zeta+\sqrt{\zeta^2-1})}\right]$$

$$Y(t) = 1 + \frac{e^{-\zeta t/\tau}}{2\sqrt{\zeta^2-1}}\left[\frac{(\zeta+\sqrt{\zeta^2-1})\, e^{\frac{\sqrt{\zeta^2-1}\,t}{\tau}} + (-\zeta+\sqrt{\zeta^2-1})\, e^{-\frac{\sqrt{\zeta^2-1}\,t}{\tau}}}{(-\zeta+\sqrt{\zeta^2-1})\,(\zeta+\sqrt{\zeta^2-1})}\right]$$

$$= 1 + \frac{e^{-\frac{\zeta t}{\tau}}}{2\sqrt{\zeta^2-1}}\left[\frac{\zeta e^{\sqrt{\zeta^2-1}\cdot\frac{t}{\tau}} - \zeta e^{-\sqrt{\zeta^2-1}\cdot\frac{t}{\tau}} + \sqrt{\zeta^2-1}\, e^{\sqrt{\zeta^2-1}\cdot\frac{t}{\tau}} + \sqrt{\zeta^2-1}\, e^{-\sqrt{\zeta^2-1}\cdot\frac{t}{\tau}}}{\zeta^2-1-\zeta^2}\right]$$

$$= 1 - \frac{e^{-\frac{\zeta t}{\tau}}}{\sqrt{\zeta^2-1}}\left[\frac{\zeta\left[e^{\sqrt{\zeta^2-1}\frac{t}{\tau}} - e^{-\sqrt{\zeta^2-1}\frac{t}{\tau}}\right]}{2} + \sqrt{\zeta^2-1}\left[\frac{e^{\sqrt{\zeta^2-1}\frac{t}{\tau}} + e^{-\sqrt{\zeta^2-1}\frac{t}{\tau}}}{2}\right]\right]$$

$$= 1 - \frac{e^{-\frac{\zeta t}{\tau}}}{\sqrt{\zeta^2-1}}\left[\zeta \sinh \frac{\sqrt{\zeta^2-1}\, t}{\tau} + \sqrt{\zeta^2-1}\,.\cosh \frac{\sqrt{\zeta^2-1}\, t}{\tau}\right]$$

$$Y(t) = 1 - e^{-\zeta\frac{t}{\tau}}\left[\frac{\zeta}{\sqrt{\zeta^2-1}} \sinh \frac{\sqrt{\zeta^2-1}\, t}{\tau} + \cosh \sqrt{\zeta^2-1}\,\frac{t}{\tau}\right]$$

where the hyperbolic functions are defined as,

$$\sinh a = \frac{e^{a} - e^{-a}}{2}, \quad \cosh a = \frac{e^{a} + e^{-a}}{2}$$

For $\zeta > 1$, the response is non-oscillatory and becomes more sluggish. This is known as an "overdamped response"

For $\zeta > 1$, the roots s_1 and s_2 are ~~used~~ real.

$$\frac{Y(s)}{X(s)} = \frac{1}{(\tau_1 s+1)(\tau_2 s+1)} \quad (A)$$

$$\frac{Y(s)}{X(s)} = \frac{1}{\tau^2 s^2 + 2\zeta\tau s + 1} \quad (A')$$

For (A') the denominator can be factored into two real linear factors.

Eqn. (A) & (A') are equivalent in this case.

By comparing the linear factors of (A) + (A')

$$\tau_1 = (\zeta + \sqrt{\zeta^2-1})\,\tau$$

$$\tau_2 = (\zeta - \sqrt{\zeta^2-1})\,\tau$$

If $\tau_1 = \tau_2$ then $\tau = \tau_1 = \tau$ and $\zeta = 1$

CASE - II STEP RESPONSE FOR ~~ζ~~ $\zeta < 1$

WKT,

$$Y(s) = \frac{1/\tau^2}{(s)(s-s_1)(s-s_2)}$$

The inversion yields,

$$Y(t) = 1 - e^{-\frac{\zeta t}{\tau}}\left[\cosh j\sqrt{1-\zeta^2}\frac{t}{\tau} + \frac{\zeta}{j\sqrt{1-\zeta^2}}\sinh j\sqrt{1-\zeta^2}\frac{t}{\tau}\right]$$

$$\cosh jx = \cos x$$
$$\sinh jx = j\sin x$$

$$Y(t) = 1 - e^{-\frac{\zeta t}{\tau}}\left[\cos\sqrt{1-\zeta^2}\frac{t}{\tau} + \frac{\zeta}{j\sqrt{1-\zeta^2}}\, j\sin\sqrt{1-\zeta^2}\frac{t}{\tau}\right]$$

For $\zeta < 1$, $\sqrt{1-\zeta^2} = p$

$$\frac{p}{\tau} = \frac{\sqrt{1-\zeta^2}}{\tau} = \omega$$

Let, $\frac{\zeta}{\sqrt{1-\zeta^2}} = P\cos\phi$, $P\sin\phi = 1$

$\phi = \tan^{-1}\left(\frac{p}{\zeta}\right)$, $\frac{P\sin\phi}{P\cos\phi} = \tan\phi = \frac{\sqrt{1-\zeta^2}}{\zeta}$

$$\phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

$$Y(t) = 1 - e^{-\zeta t/\tau}\left[\cos\omega t \cdot P\sin\phi + P\cos\phi\sin\omega t\right]$$
$$= 1 - e^{-\zeta t/\tau}\left[P\sin(\omega t + \phi)\right]$$

$$Y(t) = 1 - e^{-\zeta t/\tau}\left[P\sin\left(\frac{\sqrt{1-\zeta^2}\,t}{\tau} + \phi\right)\right]$$

$$P^2\sin^2\phi + P^2\cos^2\phi = P^2$$

$$1 + \frac{\zeta^2}{1-\zeta^2} = P^2 \qquad P = \frac{1}{\sqrt{1-\zeta^2}}$$

$$Y(t) = 1 - \frac{e^{-\frac{\zeta t}{\tau}}}{\sqrt{1-\zeta^2}} \sin\left[\frac{\sqrt{1-\zeta^2}}{\tau} t + \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right]$$

$$Y(t) = 1 - \frac{e^{-\frac{\zeta t}{\tau}}}{\sqrt{1-\zeta^2}} \sin\left[\sqrt{1-\zeta^2}\,\frac{t}{\tau} + \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right] \quad \text{—— (B)}$$

The nature of the response can be understood most clearly by plotting eqn (B) as in fig. where $Y(t)$ is plotted against the dimensionless variable t/τ for several values of ζ, including those above unity.

For $\zeta < 1$, all the response curves are oscillatory in nature and become less oscillatory as ζ is increased. The slope at the origin in fig. is zero for all values of ζ.

The response of a second order system for $\zeta < 1$ is said to be underdamped.

CASE III ; STEP RESPONSE FOR $\zeta = 1$

WKT.

$$Y(t)\Big|_{\zeta<1} = 1 - e^{-\frac{\zeta t}{\tau}}\left(\underbrace{\cos\sqrt{1-\zeta^2}\,\frac{t}{\tau}}_{1} + \frac{\zeta}{\sqrt{1-\zeta^2}}\underbrace{\sin\sqrt{1-\zeta^2}\,\frac{t}{\tau}}_{0}\right)$$

If $\zeta = 1$

$$Y(t)\Big|_{\zeta=1} = 1 - e^{-\frac{\zeta t}{\tau}}$$ practically not correct.

Use L- Hospital's rule and make $\zeta \to 1$

$$Y(t)\Big|_{\zeta=1} = 1 - e^{-\frac{\zeta t}{\tau}}\left[1 + \frac{\lim_{\zeta\to 1}\frac{d}{d\zeta}\left(\zeta \sin\sqrt{1-\zeta^2}\,\frac{t}{\tau}\right)}{\lim_{\zeta\to 1}\frac{d}{d\zeta}\left(\sqrt{1-\zeta^2}\right)}\right]$$

$$= 1 - e^{-1\cdot\frac{t}{\tau}}\left[1 + \frac{\lim_{\zeta\to 1}\left[\zeta\left(\cos\left(\sqrt{1-\zeta^2}\,\frac{t}{\tau}\right)\right)\left(\frac{t}{\tau}\right)\left(\frac{-2\zeta}{2\sqrt{1-\zeta^2}}\right) + \sin\sqrt{1-\zeta^2}\,\frac{t}{\tau}\right)}{\lim_{\zeta\to 1}\left(\frac{-2\zeta}{2\sqrt{1-\zeta^2}}\right)}\right]$$

$$= 1 - e^{-\frac{t}{\tau}}\left[1 + \lim_{\zeta\to 1}\left\{\frac{\zeta \underbrace{\cos\left(\sqrt{1-1}\,\frac{t}{\tau}\right)}_{1}\frac{t}{\tau}\left(\cancel{\frac{-2\zeta}{2\sqrt{1-\zeta^2}}}\right)}{\lim_{\zeta\to 1}\left(\cancel{\frac{-2\zeta}{2\sqrt{1-\zeta^2}}}\right)}\right\} + \frac{\lim_{\zeta\to 1}\sin\sqrt{1-\zeta^2}\,\frac{t}{\tau}}{\lim_{\zeta\to 1}\left(\frac{-2\zeta}{2\sqrt{1-\zeta^2}}\right)}\right]$$

$$Y(t)\Big|_{\zeta=1} = 1 - e^{-\frac{t}{\tau}}\left[1 + \frac{t}{\tau}\right]$$

The response is plotted in fig for $\zeta = 1$. This condition is called critically damped case. and allows most rapid approach of the response to $Y = 1$ without oscillation.

$$\frac{d}{d\zeta}\sqrt{1-\zeta^2} = \frac{1}{2}(1-\zeta^2)^{\frac{1}{2}-1}(-2\zeta)$$

$$= \frac{-2\zeta}{2\sqrt{1-\zeta^2}}$$

Terms used to describe an Underdamped System

Of these 3 cases, the underdamped response occurs most frequently in control systems. Hence a number of terms are used to describe the underdamped response quantitatively.

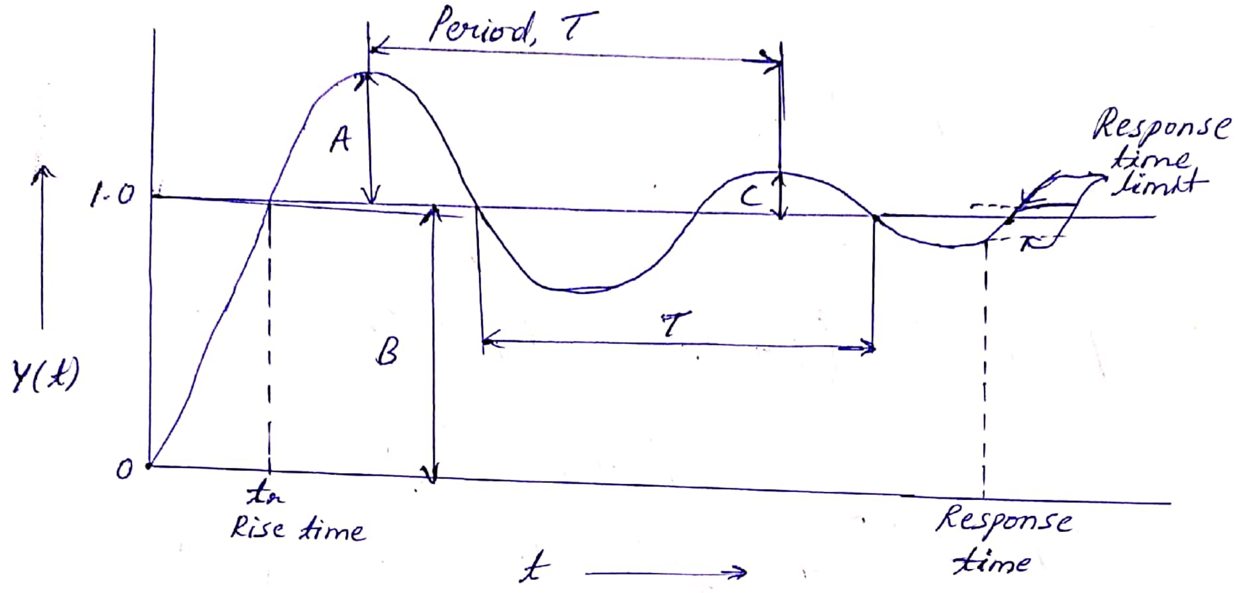


Fig - Terms used to describe an underdamped second-order response.

1. Overshoot - It is a measure of how much the response exceeds the ultimate value following a step change and is expressed as the ratio A/B in fig.

The overshoot for a unit step is related to ζ by the expression,

$$\text{Overshoot} = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right) \quad \text{--- (1)}$$

The overshoot increases for decreasing ζ.

2. Decay ratio - is defined as the ratio of the sizes of successive peaks and is given by C/A in fig.

The decay ratio is related to ζ by the expression.

$$\text{Decay ratio} = \exp\left(\frac{-2\pi\zeta}{\sqrt{1-\zeta^2}}\right) = (\text{Overshoot})^2 \quad \text{--- (2)}$$

- Larger ζ means greater damping, hence greater decay.

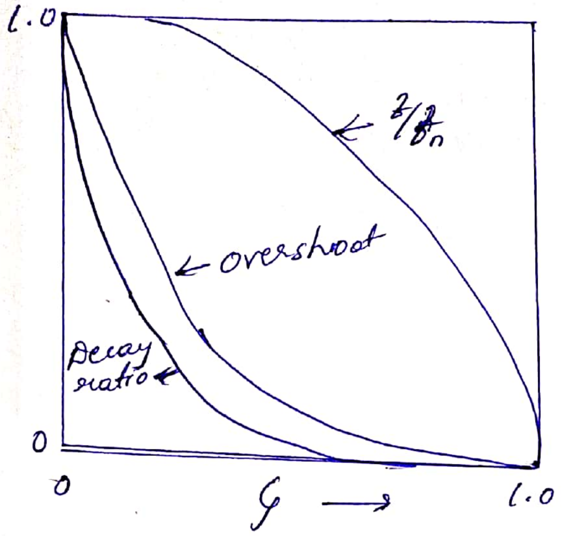


Fig - Characteristics of a step response of an underdamped second-order system.

3. Rise time — This is the time required for the response to first reach its ultimate value and is labeled t_r in fig. The t_r increases with increasing ζ.

4. Response time - This is the time required for the response to come within ± 5 percent of its ultimate value and remain there. The response time is indicated in fig.
The limits ± 5 percent are arbitrary. and other limits have been used in other texts for defining a response time.

5. Period of oscillation -
From eq.

$$Y(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\frac{\zeta t}{\tau}} \sin\left(\sqrt{1-\zeta^2}\,\frac{t}{\tau} + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

the radian frequency (radians/time) is the coefficient of t in the sine term. thus,

$$\omega, \text{radian frequency} = \frac{\sqrt{1-\zeta^2}}{\tau} \qquad \text{---} \quad (3)$$

Since the radian frequency ω is related to the ~~physical~~ cyclical frequency f by $\omega = 2\pi f$.

$$f = \frac{1}{T} = \frac{1}{2\pi}\frac{\sqrt{1-\zeta^2}}{\tau} \quad \text{——} \ (4)$$

where, T - is the period of oscillation (time/cycle). In fig, T is the time elapsed between peaks. It is also time elapsed between alternate crossings of the line $Y = 1$.

6. Natural Period of oscillation:

If the damping is eliminated [$c=0$ (or) $\zeta = 0$] in eqn.

$$W\frac{d^2Y}{dt^2} = -KY - c\frac{dY}{dt} + F(t)$$

then, the system oscillates continuously without attenuation in amplitude. Under these 'natural' (or) ~~underd~~ undamped conditions, the radian frequency is $1/\tau$ as shown by eq.

$$\omega_n = \frac{\sqrt{1-\zeta^2}}{\tau} \quad \text{——} \ (5), \text{ when } \zeta = 0$$

This frequency is referred to as the natural frequency ω_n.

$$\omega_n = \frac{1}{\tau} \quad \text{——} \ (6)$$

The corresponding natural cyclical frequency f_n and period T_n are related by the expression

$$f_n = \frac{1}{T_n} = \frac{1}{2\pi\tau} \quad \text{——} \ (7)$$

Thus, τ - has the significance of the undamped period.

From eq (4) and (7), the natural frequency is related to the actual frequency by the expression.

$$\frac{f}{f_n} = \sqrt{1-\zeta^2} \quad \text{as plotted in fig.}$$

For $\zeta < 0.5$, the natural frequency is nearly the same as the actual frequency.

It is evident that ζ - is a measure of degree of damping (or) oscillatory character and τ - is a measure of the period, (or) speed of the response of 2nd order system.

Impulse Response

If a unit impulse $\delta(t)$ is applied to the second-order system, then from eqs.

$$\frac{Y(s)}{X(s)} = \frac{1}{\tau^2 s^2 + 2\zeta\tau s + 1} \quad \text{--- (1)}$$

The transform of the response is,

$$Y(s) = \frac{1}{\tau^2 s^2 + 2\zeta\tau s + 1} \quad \text{--- (2)}$$

As in the case of step input, the nature of the response to a unit impulse will depend on whether the roots of the denominator of eq.(2) are real or complex.

Case I : Impulse Response for $\zeta < 1$

The inversion of eq.(2) for $\zeta<1$ yields the result

$\cancel{Y(t) = 1 - \frac{e^{-\zeta t/\tau}}{\sqrt{1-\zeta^2}} \sin}$ step response for $\zeta < 1$ is,

$$Y(t) = 1 - e^{-\zeta t/\tau}\left[\cos\sqrt{1-\zeta^2}\,\frac{t}{\tau} + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\sqrt{1-\zeta^2}\,\frac{t}{\tau}\right]$$

Differentiating above eqn..

$$\left.\frac{d(Y(t))}{dt}\right|_{step} = Y(t)\Big|_{impulse} = \frac{d}{dt}\left[1 - e^{-\zeta t/\tau}\left(\cos\sqrt{1-\zeta^2}\,\frac{t}{\tau} + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\sqrt{1-\zeta^2}\,\frac{t}{\tau}\right)\right]$$

$$= 0 - e^{-\zeta t/\tau}\left(-\frac{\zeta}{\tau}\right)\left[\cos\sqrt{1-\zeta^2}\,\frac{t}{\tau} + \frac{\zeta}{1-\zeta^2}\sin\sqrt{1-\zeta^2}\,\frac{t}{\tau}\right]$$

$$- e^{-\zeta t/\tau}\left[-\sin\left(\sqrt{1-\zeta^2}\,\frac{t}{\tau}\right)\left[\frac{\sqrt{1-\zeta^2}}{\tau}\right] + \frac{\zeta}{\sqrt{1-\zeta^2}}\cos\left(\sqrt{1-\zeta^2}\cdot\frac{t}{\tau}\right)\frac{\sqrt{1-\zeta^2}}{\tau}\right]$$

$$= e^{-\zeta t/\tau}\cdot\frac{\zeta}{\tau}\cdot\cos\sqrt{1-\zeta^2}\,\frac{t}{\tau} + e^{-\zeta t/\tau}\,\frac{\zeta}{\tau}\cdot\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\sqrt{1-\zeta^2}\cdot\frac{t}{\tau} + e^{-\zeta t/\tau}$$

$$\sin\sqrt{1-\zeta^2}\cdot\frac{t}{\tau}\cdot\frac{\sqrt{1-\zeta^2}}{\tau} - \cancel{e^{-\zeta t/\tau}\cdot\frac{\zeta}{\tau}\cdot\cos\sqrt{1-\zeta^2}\cdot\frac{t}{\tau}}$$

$$= \frac{e^{-\zeta t/\tau}\sin\sqrt{1-\zeta^2}}{\tau}\left(\frac{t}{\tau}\right)\left[\frac{\zeta^2}{\sqrt{1-\zeta^2}} + \sqrt{1-\zeta^2}\right] \qquad \frac{\zeta^2+1-\zeta^2}{\sqrt{1-\zeta^2}}$$

$$Y(t) = \frac{e^{-\zeta t/\tau}}{\tau\cdot\sqrt{1-\zeta^2}}\sin\sqrt{1-\zeta^2}\,\frac{t}{\tau} \quad \text{--- (3)}$$

The eqn. is plotted in the fig. The slope at the origin in fig is 1.0 for all values of ζ.

$$Y(s)\Big|_{impulse} = s \cdot Y(s)\Big|_{step} \quad \text{—— (4)}$$

The presence of s on the right side of above eqn. implies differentiation w.r.t. 't' in the time response.

In other words, the inverse transform of eq. (4) is,

$$Y(t)\Big|_{impulse} = \frac{d}{dt}\left(Y(t)\Big|_{step}\right) \quad \text{—— (5)}$$

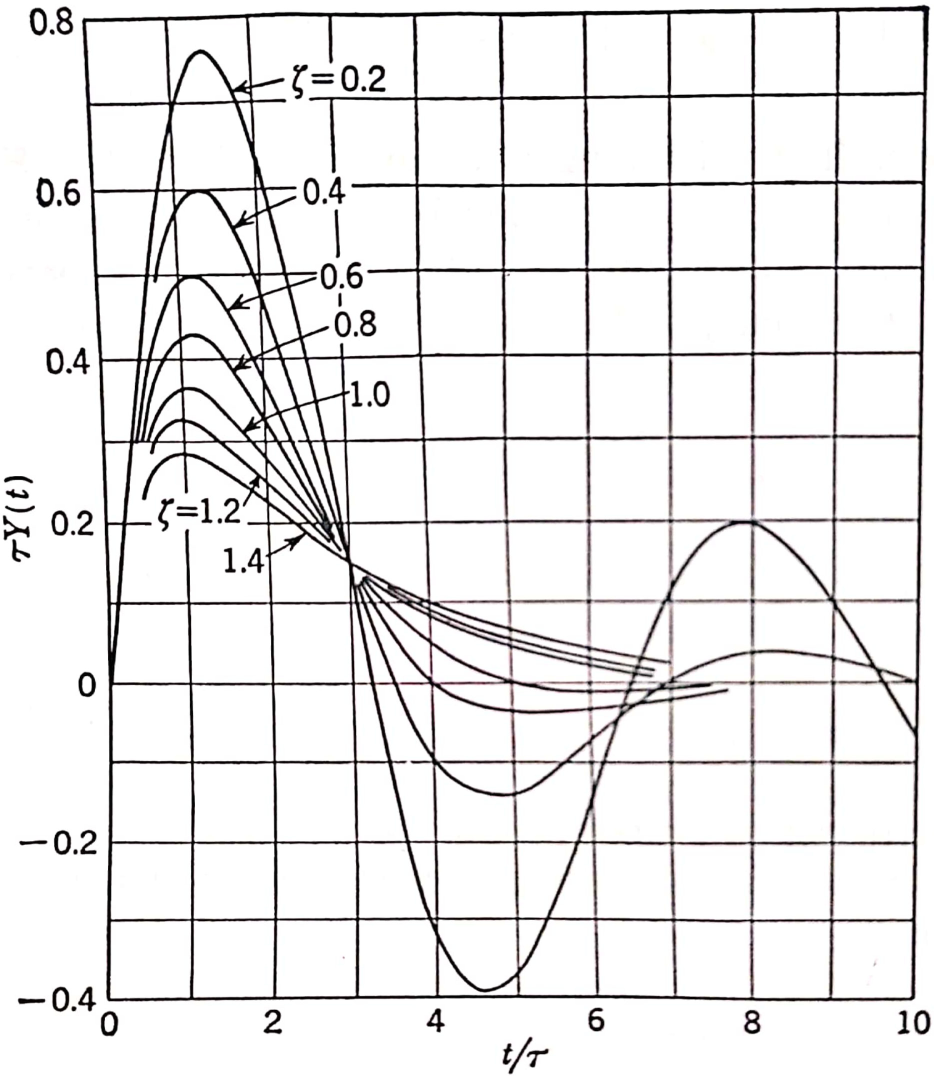


FIGURE 8-5
Response of a second-order sy
to a unit-impulse forcing func

Case II : Impulse Response for $\zeta = 1$

For the critically damped case, the response is given by

$$Y(t) = 1 - e^{-t/\tau}\left(1 + \frac{t}{\tau}\right)$$

$$Y(t)\Big|_{impulse} = \frac{d}{dt}\left[1 - e^{-t/\tau}\left(1+\frac{t}{\tau}\right)\right]$$

$$= -e^{-t/\tau}\left(1+\frac{t}{\tau}\right)\left(-\frac{1}{\tau}\right) - e^{-t/\tau}\left(0+\frac{1}{\tau}\right)$$

$$= e^{-t/\tau}\left[\frac{1}{\tau} + \frac{t}{\tau^2} - \frac{1}{\tau}\right]$$

$$Y(t)\Big|_{impulse} = \frac{t\, e^{-t/\tau}}{\tau^2} \qquad \text{—— (6)}$$

This eqn. is also plotted in fig.

Case III : Impulse response for $\zeta > 1$

For the overdamped case, the response is given by.

$$Y(t) = \frac{d}{dt}\left[1 - e^{-\frac{\zeta t}{\tau}}\left(\frac{\zeta}{\sqrt{\zeta^2-1}} \cdot \sinh\sqrt{\zeta^2-1}\,\frac{t}{\tau} + \cosh\sqrt{\zeta^2-1}\,\frac{t}{\tau}\right)\right]$$

$$= -e^{-\frac{\zeta t}{\tau}}\left(-\frac{\zeta}{\tau}\right)\left[\frac{\zeta}{\sqrt{\zeta^2-1}} \sinh\sqrt{\zeta^2-1}\,\frac{t}{\tau} + \cosh\sqrt{\zeta^2-1}\,\frac{t}{\tau}\right]$$

$$- e^{-\frac{\zeta t}{\tau}}\left[\frac{\zeta}{\sqrt{\zeta^2-1}} \cosh\sqrt{\zeta^2-1}\,\frac{t}{\tau}\left(\frac{\sqrt{\zeta^2-1}}{\tau}\right) + \sinh\sqrt{\zeta^2-1}\,.\frac{t}{\tau}.\frac{\sqrt{\zeta^2-1}}{\tau}\right]$$

$$= e^{-\frac{\zeta t}{\tau}}\frac{\zeta^2}{\tau\sqrt{\zeta^2-1}} \sinh\sqrt{\zeta^2-1}\,\frac{t}{\tau} + \cancel{e^{-\frac{\zeta t}{\tau}}\frac{\zeta}{\tau}\cosh\sqrt{\zeta^2-1}\,\frac{t}{\tau}}$$

$$- \cancel{e^{-\frac{\zeta t}{\tau}}\frac{\zeta}{\tau}\cosh\sqrt{\zeta^2-1}\,\frac{t}{\tau}} - \frac{e^{-\frac{\zeta t}{\tau}}}{\tau}\sinh\sqrt{\zeta^2-1}\,\frac{t}{\tau}\left(\sqrt{\zeta^2-1}\right)$$

$$= \frac{e^{-\frac{\zeta t}{\tau}}}{\tau}\sinh\sqrt{\zeta^2-1}\,\frac{t}{\tau}\left[\frac{\zeta^2}{\sqrt{\zeta^2-1}} - \sqrt{\zeta^2-1}\right]$$

$$Y(t) = \frac{e^{-\frac{\zeta t}{\tau}}}{\tau\sqrt{\zeta^2-1}} \sinh\sqrt{\zeta^2-1}\,\frac{t}{\tau}$$

which is plotted in fig.

To summarize, the impulse-response curves show the same general behaviour as the step-response curves. However, impulse response always turns to zero. Terms such as decay ratio, period of oscillation etc may also be used to describe the impulse response.

Eg - For stirred tank heat exchanger

Sinusoidal Response:

Consider a second order system.

$$G(s) = \frac{Y(s)}{X(s)} = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

Assume, $K = 1$.

If the forcing function is applied to the second-order system of sinusoidal nature.

$$\frac{Y(s)}{X(s)} = \frac{1}{\tau^2 s^2 + 2\tau\zeta s + 1}$$

then,

$$X(t) = A \sin \omega t \quad \text{--- (1)}$$

$$X(s) = \frac{A\omega}{s^2 + \omega^2}$$

$$\therefore \; Y(s) = \frac{A\omega/\tau^2}{(s^2+\omega^2)\left(s^2 + \frac{2\zeta s}{\tau} + \frac{1}{\tau^2}\right)} \quad \text{--- (2)}$$

The inversion of eq. (2) may be accomplished by first factoring the two quadratic terms to give

on factorising. $(s^2+\omega^2)\left(s^2 + \frac{2\zeta s}{\tau} + \frac{1}{\tau^2}\right)$

we get,

$$s_1, s_2 = \frac{-\frac{2\zeta}{\tau} \pm \frac{2}{\tau}\sqrt{\zeta^2-1}}{2} = \frac{-\zeta \pm \sqrt{\zeta^2-1}}{\tau}$$

$$s_1 = \frac{-\zeta}{\tau} + \frac{\sqrt{\zeta^2-1}}{\tau}, \quad s_2 = -\frac{\zeta}{\tau} - \frac{\sqrt{\zeta^2-1}}{\tau}$$

$$s^2 + \omega^2 = (s+j\omega)(s-j\omega)$$

$$\therefore \; Y(s) = \frac{A\omega/\tau^2}{(s-j\omega)(s+j\omega)(s-s_1)(s-s_2)} \quad \text{--- (3)}$$

$$Y(t) = \frac{A\omega}{\tau^2} L^{-1}\left[\frac{B}{(s-j\omega)} + \frac{C}{(s+j\omega)} + \frac{D}{(s-s_1)} + \frac{E}{(s-s_2)}\right]$$

$$\frac{1}{(s-j\omega)(s+j\omega)(s-s_1)(s-s_2)} = \frac{B(s+j\omega)(s-s_1)(s-s_2) + C(s-j\omega)(s-s_1)(s-s_2) + D(s-j\omega)(s+j\omega)(s-s_2) + E(s-j\omega)(s+j\omega)(s-s_1)}{(s-s_1)(s_1-s_2)(s-j\omega)(s+j\omega)}$$

$$Y(t) = \frac{A\omega}{\tau^2} L^{-1}\left[\frac{1}{(2j\omega)(j\omega-s_1)(j\omega-s_2)(s-j\omega)} + \frac{1}{(-2j\omega)(-j\omega-s_1)(-j\omega-s_2)(s+j\omega)} + \frac{1}{(s_1-j\omega)(s_1+j\omega)(s_1-s_2)(s-s_1)} + \frac{1}{(s_1-j\omega)(s_1+j\omega)(s_2-s_1)(s-s_2)}\right]$$

$$Y(t) = \frac{A\omega}{\tau^2}\left[B e^{j\omega t} + C e^{-j\omega t} + D e^{s_1 t} + E e^{s_2 t}\right]$$

$$Y(t) = \frac{A\omega}{\tau^2}\Big[B(\cos\omega t + j\sin\omega t) + C(\cos\omega t - j\sin\omega t) + De^{\left(\frac{-\zeta+\sqrt{\zeta^2-1}}{\tau}\right)t} + E\,e^{\left(\frac{-\zeta-\sqrt{\zeta^2-1}}{\tau}\right)t}\Big]$$

<u>Case I</u> : For $\zeta > 1$

$$Y(t) = \frac{A\omega}{\tau^2}\Big[\cos\omega t\,(B+C) + (B-C)j\sin\omega t + e^{-\zeta\frac{t}{\tau}}\Big[De^{\sqrt{\zeta^2-1}\frac{t}{\tau}} + Ee^{(-\sqrt{\zeta^2-1})\frac{t}{\tau}}\Big]$$

$$= \frac{A\omega}{\tau^2}\Big[F\cos\omega t + G\sin\omega t + e^{-\zeta\frac{t}{\tau}}\Big[De^{\sqrt{\zeta^2-1}\frac{t}{\tau}} + E^{(-\sqrt{\zeta^2-1})\frac{t}{\tau}}\Big]$$

ultimate response of system i.e $t \to \infty$

$$Y(t)\Big|_{\infty} = \frac{A\omega}{\tau^2}\big[F\cos\omega t + G\sin\omega t\big]$$

where, $F = B + C$

$\quad$ & $G = (B-C)j$;

As $t \to \infty$ $\quad e^{-\zeta\frac{t}{\tau}} \to 0$

$\therefore$ 3^{rd} term onwards vanishes

To find B & C, F & G.

$F = B + C$

$$F = \frac{1}{(2j\omega)(j\omega - s_1)(j\omega - s_2)} - \frac{1}{(2j\omega)(j\omega + s_1)(j\omega + s_2)}$$

$$F = \frac{s_1 + s_2}{(j\omega - s_1)(j\omega - s_2)(j\omega + s_1)(j\omega + s_2)}$$

$$G = (B-C)j = \frac{1}{\omega}\left[\frac{s_1 s_2 - \omega^2}{(j\omega - s_1)(j\omega - s_2)(j\omega + s_1)(j\omega + s_2)}\right]$$

But, $p\cos A + q\sin A = r\sin(A + \theta)$

$r = \sqrt{p^2 + q^2}$

$\theta = \tan^{-1}\left(\frac{p}{q}\right)$

$$Y(t)\Big|_{t\to\infty} = \frac{A\omega}{\tau^2}\Big[\sqrt{F^2 + G^2}\Big]\sin[\omega t + \phi]$$

where, $\phi = \tan^{-1}\frac{F}{G}$

$$\sqrt{F^2+G^2}=\sqrt{\frac{(s_1+s_2)^2}{(j\omega-s_1)^2(j\omega+s_1)^2(j\omega-s_2)^2(j\omega-s_2)^2}-\frac{(s_1s_2-\omega^2)^2}{\omega^2[(j\omega-s_1)^2(j\omega-s_2)^2(j\omega+s_1)^2(j\omega+s_2)^2]}}$$

$$s_1+s_2=\frac{-2\zeta}{\tau}\quad , \quad s_1s_2=\frac{1}{\tau^2}$$

$$\sqrt{F^2+G^2}=\frac{1}{(j\omega-s_1)(j\omega+s_1)(j\omega+s_2)(j\omega-s_2)}\sqrt{\frac{4\zeta^2}{\tau^2}+\frac{(1-(\omega\tau)^2)^2}{\tau^4\omega^2}}$$

$$=\frac{1}{\tau^2\omega\left[\omega^4+\omega^2\left(\frac{4\zeta^2}{\tau^2}-\frac{2}{\tau^2}\right)\right]+\frac{1}{\tau^4}}\sqrt{(1-(\omega\tau^2)^2+(2\zeta\tau\omega)^2}$$

$$\therefore Y(t)\Big|_{\infty}=\left(\frac{A\omega}{\tau^2}\right)\left(\frac{1}{\tau^2\omega\left(\omega^4+\omega^2\left(\frac{4\zeta^2}{\tau^2}-\frac{2}{\tau^2}\right)\right)+\frac{1}{\tau^4}}\right)\left(\sqrt{(1-(\omega\tau)^2)^2+(2\zeta\tau\omega)^2}\right).\sin(\omega t+\phi)$$

$$=\frac{A}{[(\omega\tau)^4+4\omega^2\tau^2\zeta^2-2\omega^2\tau^2+1]}\left(\sqrt{(1-(\omega\tau)^2)^2+(2\zeta\tau\omega)^2}\right)\sin(\omega t+\phi)$$

$$=\frac{A}{[(\omega\tau)^4+4\omega^2\tau^2\zeta^2-2\omega^2\tau^2+1]}\cdot\sqrt{(1-(\omega\tau)^2)^2+(2\zeta\tau\omega)^2}\cdot\sin(\omega t+\phi)$$

$$Y(t)\Big|_{\infty}=\frac{A\cdot\sin(\omega t+\phi)}{\sqrt{(1-(\omega\tau)^2)^2+(2\zeta\omega\tau)^2}}\qquad\text{——— (4)}$$

where, $\phi=\tan^{-1}\frac{F}{G}=\tan^{-1}\frac{2\zeta\omega\tau}{1-(\omega\tau)^2}$

Case II : For $\zeta<1$

$$Y(t)=\frac{A\omega}{\tau^2}\left[\cos\omega t\,(B+C)+(B-C)\,j\sin\omega t+e^{-\zeta\frac{t}{\tau}}\left[D\,e^{j\sqrt{1-\zeta^2}\frac{t}{\tau}}+(\tau-e^{(-i\sqrt{1-\zeta^2})t/\tau}\right]\right]$$

As $t\to\infty$, 3rd term onwards vanishes.

$$\therefore Y(t)\Big|_{\infty}=\frac{A\omega}{\tau^2}[F\cos\omega t+G\sin\omega t]$$

$F=B+C$, and $G=(B-C)i$

$$F=B+C=\frac{s_1+s_2}{(j\omega+s_1)(j\omega-s_1)(j\omega-s_2)(j\omega+s_2)}$$

$$G = \frac{1}{\omega}\left[\frac{s_1 s_2 - \omega^2}{(j\omega - s_1)(j\omega - s_2)(j\omega + s_1)(j\omega + s_2)}\right]$$

$$s_1 = \frac{-\zeta + \sqrt{\zeta^2 - 1}}{\tau} \quad \text{for } \zeta < 1$$

$$s_1' = \frac{-\zeta + i\sqrt{1 - \zeta^2}}{\tau}$$

Similarlly,

$$s_2 = \frac{-\zeta - \sqrt{\zeta^2 - 1}}{\tau}$$

$$s_2' = \frac{-\zeta - j\sqrt{1 - \zeta^2}}{\tau}$$

$$F = \frac{s_1' + s_2'}{(j\omega + s_1')(j\omega - s_1')(j\omega + s_2')(j\omega - s_2')}$$

$$s_1' + s_2' = -\frac{2\zeta}{\tau}$$

$$\therefore G = \frac{1}{\omega}\left[\frac{s_1' s_2' - \omega^2}{(j\omega - s_1')(j\omega - s_2')(j\omega + s_1')(j\omega + s_2')}\right]$$

$$s_1' s_2' = \frac{1}{\tau^2}$$

$$\sqrt{F^2 + G^2} = \frac{1}{\tau^2\omega\left[\omega^4 + \omega^2\left(\frac{4\zeta^2}{\tau^2} - \frac{2}{\tau^2}\right) + \frac{1}{\tau^4}\right]} \sqrt{[1 - (\omega\tau)^2]^2 + (2\zeta\tau\omega)^2}$$

$$\therefore Y(t)\Big|_{\infty} = \frac{A}{\sqrt{(1 - (\omega\tau)^2)^2 + (2\zeta\omega\tau)^2}} \cdot \sin(\omega t + \phi) \quad \text{--- (5)}$$

$$\phi = -\tan^{-1}\frac{2\zeta\omega\tau}{1 - (\omega\tau)^2} \quad \text{and} \quad \phi = \tan^{-1}\left(\frac{F}{G}\right)$$

Case III ; For $\zeta = 1$

Consider $G(s) = \dfrac{1}{\tau^2 s^2 + 2\zeta\tau s + 1}$

$$\zeta = 1, \quad G(s) = \frac{1}{\tau^2 s^2 + 2\tau s + 1}$$

$$\frac{Y(s)}{X(s)} = \frac{1}{\tau^2\left(s + \frac{1}{\tau}\right)^2}$$

$$Y(s) = \frac{A\omega}{\tau^2(s^2+\omega^2)} \cdot \frac{1}{(s+\frac{1}{\tau})^2}$$

$$\frac{1}{(s^2+\omega^2)(s+\frac{1}{\tau^2})} = \frac{B}{(s-j\omega)} + \frac{C}{(s+j\omega)} + \frac{D}{(s+\frac{1}{\tau})} + \frac{E}{(s+\frac{1}{\tau})^2}$$

$$Y(s) = \frac{A\omega}{\tau^2}\left[\frac{B}{(s-j\omega)} + \frac{C}{(s+j\omega)} + \frac{D}{(s+\frac{1}{\tau})} + \frac{E}{(s+\frac{1}{\tau})^2}\right]$$

$$Y(t) = \frac{A\omega}{\tau^2}\left[B.e^{j\omega t} + Ce^{-j\omega t} + De^{-t/\tau} + Ete^{-t/\tau}\right]$$

$$Y(t) = \frac{A\omega}{\tau^2}\left[B.e^{j\omega t} + Ce^{-j\omega t} + e^{-t/\tau}(D+Et)\right]$$

If $t \to \infty$, then 3rd term vanishes

$$\therefore Y(t)\Big|_{\infty} = \frac{A\omega}{\tau^2}\left[Be^{j\omega t} + Ce^{-j\omega t}\right]$$

$$= \frac{A\omega}{\tau^2}\left[(B+C)\cos\omega t + (B-C)\,i\sin\omega t\right]$$

$$Y(t)\Big|_{\infty} = \frac{A\omega}{\tau^2}\left[F\cos\omega t + G\sin\omega t\right]$$

$F = B+C$ and $G = (B-C)j$

$$F = \frac{1}{2j\omega(j\omega+\frac{1}{\tau})^2} - \frac{1}{2j\omega(-j\omega+\frac{1}{\tau})^2}$$

$$= \frac{-2}{\tau(j\omega+\frac{1}{\tau})^2(\frac{1}{\tau}-j\omega)^2}$$

$$G = \frac{2(\frac{1}{\tau^2}-\omega^2)}{2j\omega(j\omega+\frac{1}{\tau})^2(\frac{1}{\tau}-j\omega)^2}$$

$$G = \frac{(\frac{1}{\tau^2}-\omega^2)}{\omega(j\omega+\frac{1}{\tau})^2(\frac{1}{\tau}-j\omega)^2}$$

$$\sqrt{F^2+G^2} = \frac{1\ \sqrt{(1-(\omega\tau)^2)+(2\tau\omega)^2}}{\tau^2\omega(\omega^4+\omega^2(\frac{4}{\tau^2}-\frac{2}{\tau^2})+\frac{1}{\tau^4})} \cdot \cancel{\sin(\omega t+\phi)}$$

For $G = 1$

$$Y(t)\Big|_{\infty} = \frac{A}{\sqrt{(1-(\omega\tau)^2)^2+(2\omega\tau)^2}}\sin(\omega t+\phi) \quad \text{—— (6)}$$

where, $\phi = \tan^{-1}\frac{F}{G} = -\tan^{-1}\left(\frac{2\omega\tau}{(1-(\omega\tau)^2)}\right)$

it is observed that;

1. The ratio of the output amplitude to the input amplitude is

$$\frac{1}{\sqrt{[1-(\omega\tau)^2]^2+(2\zeta\omega\tau)^2}}$$

greater or less than 1, depending on ζ and $\omega\tau$. This is in direct contrast to the sinusoidal response of the first-order system, where ratio of the output amplitude to the input amplitude is always less than 1.

2. The output lags the input by phase angle $|\phi|$. ~~It can be~~ Eqn. 6 shows that $|\phi|$ approaches $180°$ asymptotically as ω increases. The phase lag of the first-order system, on the other hand, can never exceed $90°$.

Transportation Lag

Also known as 'dead time' and distance velocity lag.

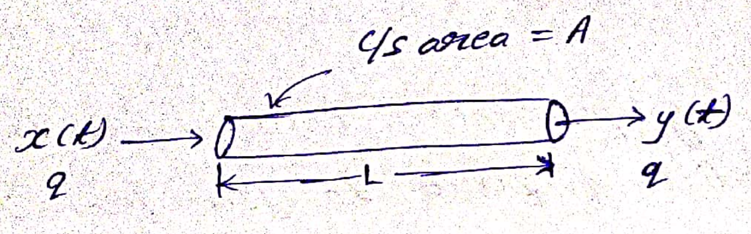


Fig- System with transportation lag.

A liquid flows through an insulated tube of uniform cross-sectional area A and length L at a constant volumetric flow rate q. The density ρ and the heat capacity C are constant. The tube wall has negligible heat capacity and the velocity profile is flat (plug flow).

The temp. x of the entering fluid varies with time, and it is desired to find the response of the outlet temp. equals the outlet temp. y(t) in terms of a transfer function.

$$\cancel{x_s = y_s} \quad \text{———} \; ①$$

It is assumed that the system is initially at steady state, i.e. the inlet temp equals the outlet temp i.e.

$$x_s = y_s \quad \text{———} \; ①$$

If a step change is made in x(t) at t=0, the change would not be detected at the end of the tube until τ sec. later, where τ is the time required for the entering fluid to pass through the tube.

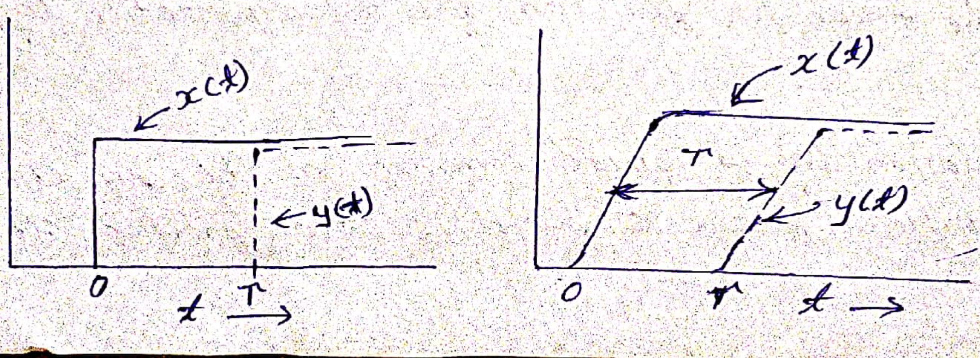


Fig- Response of transportation lag to various inputs

If the variation in $x(t)$ were some arbitrary function as in fig (b) the response $y(t)$ at the end of the pipe would be identical with $x(t)$ but again delayed by τ units of time.

The transportation lag parameter τ is simply the time needed for a particle of fluid to flow from the entrance of the pipe to the exit, and is calculated as,

$$\tau = \frac{\text{volume of pipe}}{\text{volumetric flow rate}}$$

$$\tau = \frac{Al}{q} \quad \text{———} \quad (2)$$

From fig. the relationship between $y(t)$ and $x(t)$ is

$$y(t) = x(t-\tau) \quad \text{———} \quad (3)$$

Subtracting eq. (1) from (3) and introducing deviation variables $X = x - x_s$

$Y = y - y_s$ gives

$$Y(t) = X(t-\tau) \quad \text{———} \quad (4)$$

If the Laplace transform of $X(t) = X(s)$

Laplace transform of $X(t-\tau) = e^{-s\tau} X(s)$

From theorem on translation of a function, eq (4) becomes

$$Y(s) = e^{-s\tau} X(s)$$

$$\frac{Y(s)}{X(s)} = e^{-s\tau} \quad \text{———} \quad (5)$$

Therefore, the transfer function of a transportation lag is $e^{-s\tau}$.

The transportation lag is quite common in the ~~chm~~ chemical process industries where a fluid is transported through a pipe.

Approximation of Transport Lag

The transport lag is quite different from the other transfer functions (first-order, second-order etc).. A system containing a transport lag cann't be analyzed for stability by the Routh test.

One approach to approximating the transport lag is to write $e^{-\tau s}$ as $\frac{1}{e^{\tau s}}$ and to express the denominator as a Taylor series. The result is,

$$e^{-\tau s} = \frac{1}{e^{\tau s}} = \frac{1}{1 + \tau s + \frac{\tau^2 s^2}{2} + \frac{\tau^3 s^3}{3!} + \cdots\cdots}$$

Keeping only the first two terms in the denominator gives,

$$e^{-\tau s} \simeq \frac{1}{1 + \tau s}$$

This approximation, which is simply a first-order lag, is a crude approximation of a transport lag.

$$e^{-\tau s} = \frac{e^{-\tau s/2}}{e^{\tau s/2}}$$

Expanding numerator and denominator in a Taylor series and keeping only terms of first-order give.

$$e^{-\tau s} \simeq \frac{1 - \tau s/2}{1 + \tau s/2} \quad 1^{st}\text{-order approximation.}$$

This expression is also known as 1^{st}-order pade approximation Approximation for a transport lag is the 2nd-order pade approximation.

$$e^{-\tau s} \simeq \frac{1 - \tau s/2 + \tau^2 s^2/12}{1 + \tau s/2 + \tau^2 s^2/12} \quad \text{— } 2^{nd}\text{-order pade.}$$

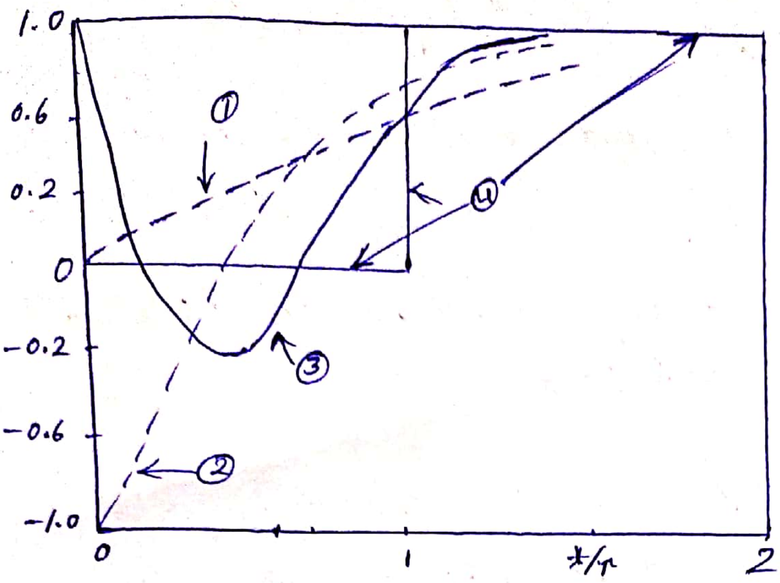


Fig - Step response to approximations of the transport lag $e^{-T's}$
1. $1/(T's+1)$
2. 1st order pade
3. 2nd order pade
4. $e^{-T's}$

The step responses of the three approximations of transport lag presented in fig. The step response of $e^{-T's}$ is also shown for comparison. The response for the first-order pade approximation drops to -1 before rising exponentially toward +1. The response for the second-order pade approximation jumps to +1 and then descends to below zero before returning gradually back to +1.

No approximations for $e^{-T's}$ is very accurate, the approximation for $e^{-T's}$ is more useful when it is multiplied by several first-order or second-order transfer functions. In this case, the other transfer functions filter out the high frequency content of the signals passing through the transport lag with the result that the transport lag approximation, when combined with other transfer functions, provides a satisfactory result in many cases. The accuracy of a transport lag can be evaluated. most clearly in terms of frequency response.

Aruchhya

FIFTEEN

Additional Solved Examples

First order System

Example 1 : A thermometer having first order dynamics with a time constant of 6 second is placed in a temperature bath at 30°C. After the thermometer reaches steady state it is suddenly placed in a hot fluid at 60°C.

Calculate the temperature indicated by the thermometer at t = 3 sec, t = 6 sec and t = 10 sec.

Solution : The thermometer is subjected to a step change of magnitude

$$M = (60-30) = 30^0C$$

The steady state temperature of the bath

$$x_s = 30^0C$$

The steady state temperature of the thermometer

$$y_s = 30^0C$$

The step response equation of thermometer having first order dynamics can be written as

$$Y(t) = M\ (1-e^{-t/\tau})$$

Where,

$$Y(t) = y(t) - y_s$$

y (t) = Temperature indicated by thermometer at t = t

y_s = Steady state temperature

The temperature indicated by the thermometer can be obtained as

$$y(t) = y_s + M\ (1-e^{-t/\tau})$$

For time t = 3 sec

$$y(t) = 30 + 30\ (1-e^{-3/6})$$

$$y(t) = 30+30\times 0.3934$$

$$y(t) = 41.8^0C$$

For time t = 6 sec

$y(t) = 30+$

$y(t) = 30+$

$y(t) = 48.9$

For time t = 10 se

$y(t) = 30$

$y(t) = 30$

$y(t) = 54.3$

The temperature

$y(t) = 41.8$

$y(t) = 48.9$

$y(t) = 54.3$

Example 2 : A th

time constant of 6 secon

magnitude 3°C. The stea

the temperature indica

second.

Solution : The r

is written as

$Y(t) = \frac{A}{\tau}$

Where,

A is the m

$Y(t) = y$

y (t) = Tempera

y_s = Steady stat

The temperature

$y(t) = y_s$

$y(t) = 30$

For time t = 3 s

$y(t) = 30$

$y(t) = 30$

Scanned with CamScanner

For time t = 6 sec

$$y(t) = 30 + 30(1 - e^{-6/6})$$

$$y(t) = 30 + 18.9$$

$$y(t) = 48.9^{\circ}C$$

For time t = 10 sec

$$y(t) = 30 + 30\ (1 - e^{-10/6})$$

$$y(t) = 30 + 24.33$$

$$y(t) = 54.33^{\circ}C$$

The temperature indicated by the thermometer

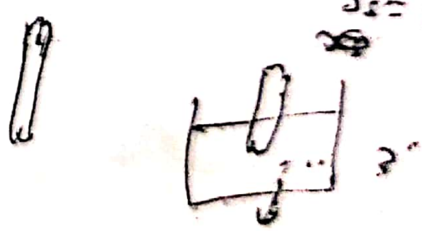
$y(t) = 41.8^{\circ}C$ at $t = 3$ sec
$y(t) = 48.9^{\circ}C$ at $t = 6$ sec
$y(t) = 54.33^{\circ}C$ at $t = 10$ sec.

Example 2 : A thermometer is observed to exhibit the first order dynamics is having time constant of 6 second is placed in a bath. The bath is subjected to the impulse change of magnitude 3°C. The steady state temperature indicated by the thermometer is 30°C. Calculate the temperature indicated by the thermometer at t = 3 second, t = 6 second and t = 18 second.

Solution : The response equation for the thermometer subjected to impulse change is written as

$$Y(t) = \frac{A}{\tau} e^{-t/\tau}.$$

$\tau = 6$ Sec



$y_s = 3c$

$x = 30°$

Where,

A is the magnitude of impulse change which is 3°C

$$Y(t) = y(t) - y_s$$

$y(t)$ = Temperature indicated by thermometer at t = t
y_s = Steady state temperature
The temperature indicated by the thermometer is expressed as

$$y(t) = y_s + \frac{A}{\tau} e^{-t/\tau}.$$

$$y(t) = 30 + \frac{3}{6} e^{-t/6}$$

For time t = 3 second

$$y(t) = 30 + 0.5\,e^{-3/6}$$

$$y(t) = 30 + 0.303 = 30.303^{\circ}C$$

For time t = 6 second

$$y(t) = 30 + 0.5\, e^{-6/6}$$

$$y(t) = 30.18^0 C$$

For time t = 18 second

$$y(t) = 30 + 0.5\ e^{-18/6}$$

$$y(t) = 30.024^0 C.$$

The temperature indicated by the thermometer

$y(t) = 30.303°C$ at $t = 3$ sec

$y(t) = 30.18°C$ at $t = 6$ sec

$y(t) = 30.024°C$ at $t = 18$ sec.

Example 3 : A thermometer having first order dynamics with time constant of 9 second is placed in a temperature bath at 50°C. After the thermometer reaches steady state temperature of 50°C with the bath, the bath temperature is varied linearly with time at the rate of 5°C/min.

Calculate the temperature indicated by the thermometer at t = 3 second, t = 6 second and t = 19 second.

Solution : The thermometer is subjected to linear change. The response equation of first order system subjected to the linear change is written as

$$Y(t) = A(t - \tau) + A\tau e^{-t/\tau}$$

Where,

'A' is the slope of the variation

$$A = \frac{5}{60}\ °C/sec$$

$$A = 0.0833°\ C/sec$$

The steady state temperature of the thermometer is 50°C

$$y_s = 50^0 C$$

and $Y(t) = y(t) - y_s$

The temperature indicated by the thermometer is written as

$$y(t) = y_s + A(t - \tau) + A\tau e^{-t/\tau}.$$

$y(t)$ = Temperature indicated by thermometer at time t = t

y_s = Steady state temperature

A = Slope of variation

For time t = 3 second

$$y(t) = 50 + 0.0833\left[(3-9) + 9e^{-3/9}\right]$$

$$y(t) = 50 + 0.03 = 50.03^0C$$

For time t = 6 second

$$y(t) = 50 + 0.0833\left[(6-9) + 9e^{-6/9}\right]$$

$$y(t) = 50 + 0.1 = 50.1^0C$$

For time t = 19 second

$$y(t) = 50 + 0.0833\left[(19-9) + 9e^{-19/9}\right]$$

$$y(t) = 50 + 0.9 = 50.9^0C$$

The temperature indicated by the thermometer

y (t) = 50.03°C at t = 3 sec

y (t) = 50.1°C at = t = 6 sec

y (t) = 50.9°C at t = 19 sec.

Example 4 : A thermometer having time constant 5 second is at steady state temperature of 30°C and is suddenly immersed in a bath of 60°C. Determine the time to attain 55°C by the thermometer.

Solution : The response equation of thermometer for step change is expressed as

$$Y(t) = M(1 - e^{-t/\tau})$$

The magnitude of step change is

$$M = (60 - 30) = 30^0C$$

Steady state temperature of themometer is 30°C

$$Y(t) = y(t) - y_s$$

The temperature indicated by the thermometer is expressed as

$$y(t) = y_s + M(1 - e^{-t/\tau})$$

$$y(t) = 30 + 30(1 - e^{-t/5})$$

For $y(t) = 55^0C$

$$55 = 30 + 30(1 - e^{-t/5})$$

$$(1 - e^{-t/5}) = \left(\frac{55-30}{30}\right)$$

$$\tau s\ (1-e^{-t/5}) = 0.833$$

$$e^{-t/5} = 1-0.833$$

$$e^{-t/5} = 0.1666$$

Take log on both side, $\log_e e^{-t/5} = \log_e 0.1666$

$t = 9$ sec.

$-t/5 = -1.792 = 8.96079$

Example 5 : A thermometer having time constant of 6 second is placed in a temperature bath which is at 50°C. After the thermometer reaches steady state temperature of 50°C with the bath, the bath temperature is subjected to impulse change of magnitude 5°C.

Calculate the temperature indicated by the thermometer at t = 5 second, t = 10 second and t = 19 second.

Solution : The impulse response equation of thermometer for first order system is expressed as

$$Y(t) = \frac{A}{\tau} e^{-t/\tau}$$

Where,

$$Y(t) = y(t) - y_s$$

A = magnitude of impulse change

The temperature indicated by the thermometer is expressed as

$$y(t) = y_s + \frac{A}{\tau} e^{-t/\tau}$$

Magnitude of impulse change A = 5°C

For time t = 5 second

$$y(t) = 50 + \frac{5}{6} e^{-5/6}$$

$$y(t) = 50 + 0.36 = 50.36^0C$$

For time t = 10 second

$$y(t) = 50 + \frac{5}{6} e^{-10/6}$$

$$y(t) = 50 + 0.15 = 50.15^0C$$

For time t = 19 second

$$y(t) = 50 + \frac{5}{6} e^{-19/6}$$

$$y(t) = 50 + 0.035 = 50.035^0C$$

The temperature indicated by the thermometer

$y(t) = 50.36°C$ at $t = 5$ sec

$y(t) = 50.15°C$ at $t = 10$ sec

$y(t) = 50.035°C$ at $t = 19$ sec.

Example 6 : A thermometer is observed to exhibit the first order dynamics. The response of the thermometer was evaluated by applying a step change in the bath in which the thermometer is placed. After 6 second the output value was 63.2 percent of the total applied step change. The thermometer reading at temperature of 55°C was put into a bath whose temperature was varied sinusoidally between 50°C and 60° with a 60 sec/cycle period of oscillation.

Determine :

(i) Maximum and minimum temperature indicated by the thermometer

(ii) Phase lag

(iii) Amplitude ratio.

Solution : The response equation of the first order system subjected to sinusoidal change at steady state is expressed as

$$Y(t) = \frac{A}{\sqrt{1+(w\tau)^2}} \sin(wt + \theta)$$

Where,

$$\theta = \tan^{-1}(-w\tau)$$

Input amplitude of variation $A = 5°C$

Output amplitude of variation

$$= \frac{A}{\sqrt{1+(w\tau)^2}}$$

Where,

w = radian frequency

$w = 2\pi f.$

Period of oscillation $T = \frac{1}{f}$. $f = \frac{1}{T}$

$\omega = 2\pi f$

f = cyclical frequency

$\omega = 2\pi \frac{1}{T}$

$$w = \frac{2\pi}{T} = \frac{2 \times 3.14}{60}$$

$w = 0.104$ rad/sec

Radian frequency $w = 0.104$ rad/sec.

The response of the first order control system becomes 63.2 percent of the total applied magnitude of the step change for $\frac{t}{\tau} = 1$ or the time elapsed is equal to one time constant for the step change.

Therefore the time constant of the thermometer $\tau = 6$ second

$$\text{Amplitude ratio} = \frac{1}{\sqrt{1+(w\,\tau)^2}}$$

$$= \frac{1}{\sqrt{1+(0.104\times 6)^2}} = 0.8$$

Output amplitude of variation

$$= \frac{A}{\sqrt{1+(w\,\tau)^2}}$$

$$= \frac{5}{\sqrt{1+(0.104\times 6)^2}}$$

$$= 4.24°C$$

The temperature is oscillating about the average temperature of 55°C

The minimum temperature indicated by the thermometer

$$= (55 - 4.24) = 50.76°C$$

The maximum temperature indicated by the thermometer

$$= (55 + 4.24) = 59.24°C$$

The phase lag is expressed as

$$\theta = \tan^{-1}(-w\,\tau)$$

$$\theta = \tan^{-1}(-0.104\times 6)$$

$$\theta = \tan^{-1}(0.624)$$

$$\theta = -31.9^0$$

$$\text{Phase lag} = \frac{|\theta|\,T}{360}$$

$$= \frac{31.9\times 60}{360} = 5.31 \text{ sec.}$$

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Example 7 : A thermometer having first order dynamics is placed in a bath of temperature 50°C. After the thermometer reaches steady state temperature with the bath. The bath is subjected to step change of magnitude 30°C. The time constant of thermometer is 6 second. Determine the temperature indicated by the thermometer at t = 6 second.

Solution : The response equation of first order control system is expressed as

$$Y(t) = M\,(1-e^{-t/\tau})$$

Magnitude of step change M = 30°C

$$Y(t) = y(t) - y_s$$

The temperature indicated by the thermometer is expressed as

$$y(t) = y_s + M(1-e^{-t/\tau})$$

$$y(t) = 50+30(1-e^{-6/6})$$

$$y(t) = 68.91^0C.$$

Example 8 : A temperature alarm unit exhibit the first order dynamics having time constant 90 second is subjected to 90 K rise because of fire. The increase of 30 K is needed to respond the alarm.

Calculate the time needed for signalling the temperature change.

Solution : A temperature alarm unit is subjected to step change of magnitude 90 K

The step response equation for first order system is expressed as

$$Y(t) = M(1-e^{-t/\tau})$$

The response of the alarm unit Y (t) is 30 K

$$30 = 90\,(1-e^{-t/90})$$

$$e^{-t/9} = 0.666$$

$$t = 36.5 \text{ sec.}$$

Example 9 : The hot junction of a thermocouple is at temperature of 30°C. The thermocouple is placed in a bath at 80°C. Calculate the time needed to attain 63.2 percent of the total applied magnitude of step change. The time constant of thermocouple is 6 sec.

Solution : The response equation of the first order control system is expressed as

$$Y(t) = M\,(1-e^{-t/\tau})$$

Magnitude of step change

$$M = (80-30) = 50°C$$

The response of thermocouple is expressed as

$$Y(t) = 0.632 \times 50$$

$$Y(t) = 31.6°C$$

$$31.6 = 50\,(1 - e^{-t/\tau})$$

$$e^{-t/\tau} = 0.368$$

$$\frac{t}{\tau} = 0.999$$

$$t = 0.999 \times 6 = 5.99 \text{ sec}$$

Example 10 : A thermometer having first order dynamics is placed in a bath. The time constant of thermometer is 9 second. The bath is subjected to step change of magnitude 30°C.

Calculate the time needed to reach 63.2 percent of the total applied magnitude of step change.

Solution : The response equation of the first order control system is expressed as

$$Y(t) = M\,(1 - e^{-t/\tau})$$

Magnitude of step change $M = 30°C$

$$Y(t) = 0.632 \times 30 = 18.9^0C$$

$$18.9 = 30\,(1 - e^{-t/\tau})$$

$$\frac{t}{\tau} = 1$$

$$t = 9 \text{ sec.}$$

Example 11 : A tank having a cross sectional area of 0.3 m^2. The steady state flowrate of water in the tank is $0.5 \text{ m}^3/\text{min}$ is suddenly increased to $0.55 \text{ m}^3/\text{min}$. The level in the tank is 0.6 m for $0.06 \text{ m}^3/\text{min}$ flowrate and 0.5 m for $0.03 \text{ m}^3/\text{min}$ flowrate. Determine the outlet flowrate at t = 0.5 min, t = 1 min and t = 1.5 min.

Solution : The transfer function of liquid level tank is expressed as

$$\frac{Q_0(s)}{Q_i(s)} = \frac{1}{\tau s + 1}$$

The resistance R is expressed as

$$R = \frac{\Delta H}{\Delta Q_0}$$

$$R = \frac{0.6 - 0.5}{0.06 - 0.03} = 3.33$$

The time constant for the tank is expressed as

$$\tau = AR$$

$$\tau = 0.3 \times 3.33 = 1 \text{ min}$$

The response equation for first order control system for step change is expressed as

$$Q_0(t) = M(1-e^{-t/\tau})$$

Where,

$$Q_0(t) = q_0(t) - q_{0s}$$

The output flowrate is expressed as

$$q_0(t) = q_{0s} + M(1-e^{-t/\tau})$$

q_{0s} = Steady state flowrate

M = Magnitude of step change

$q_0(t)$ = Flowrate of water at time $t = t$

τ = Time constant for tank

Magnitude of step change

$$M = (0.55 - 0.5) = 0.05 \text{ m}^3/\text{min}$$

The output flowrate is expressed as

$$q_0(t) = 0.5 + 0.05(1-e^{-t})$$

For time $t = 0.5$ min

$$q_0(t) = 0.5 + 0.05(1-e^{-0.5})$$

$$q_0(t) = 0.51 \text{ m}^3/\text{min}$$

For time $t = 1$ min

$$q_0(t) = 0.5 + 0.05(1-e^{-1})$$

$$q_0(t) = 0.53 \text{ m}^3/\text{min}$$

For time $t = 1.5$ min

$$q_0(t) = 0.5 + 0.05(1-e^{-1.5})$$

$$q_0(t) = 0.538 \text{ m}^3/\text{min}.$$

Example 12 : A tank having cross sectional area of 0.6 m². The steady state flowrate of water is 0.5 m³/min. The flowrate is subjected to step change of magnitude 0.06 m³/min. The time constant for the tank is 1.5 min. Determine the outlet flowrate of water at $t = 1$ min, $t = 1.5$ min, $t = 5$ min and $t = 9$ min.

Solution : The response equation for the tank subjected to step change is expressed as

$$Q_0(t) = M(1-e^{-t/\tau})$$

Where,

$$Q_0(t) = q_0(t) - q_{0s}$$

$q_0(t)$ = output flowrate of water at time t = t

q_{0s} = output flowrate of water at steady state

M = magnitude of step change

τ = time constant

The outlet flowrate of water in the tank is expressed as

$$q_0(t) = q_{0s} + M(1 - e^{-t/\tau})$$

For time t = 1 min

$$q_0(t) = 0.5 + 0.06\ (1 - e^{-1/1.5})$$

$$q_0(t) = 0.53\ m^3/min$$

For time t = 1.5 min

$$q_0(t) = 0.5 + 0.06\ (1 - e^{-1.5/1.5})$$

$$q_0(t) = 0.538\ m^3/min$$

For time t = 5 min

$$q_0(t) = 0.5 + 0.06\ (1 - e^{-5/1.5})$$

$$q_0(t) = 0.55\ m^3/min$$

For time t = 9 min

$$q_0(t) = 0.5 + 0.06\ (1 - e^{-9/1.5})$$

$$q_0(t) = 0.559\ m^3/min$$

The outlet flowrate of water is

$$q_0(t) = 0.53\ m^3/min \text{ for } t = 1 \text{ min}$$

$$q_0(t) = 0.538\ m^3/min \text{ for } t = 1.5 \text{ min}$$

$$q_0(t) = 0.55\ m^3/min \text{ for } t = 5 \text{ min}$$

$$q_0(t) = 0.559\ m^3/min \text{ for } t = 9 \text{ min}$$

Example 13 : A tank having cross sectional area of 0.5 m^2. The steady state flowrate of liquid is 0.6 m^3/min. The flowrate is subjected to the impulse change of magnitude 0.05 m^3/min. The time constant for the tank is 1 min. Determine the outlet flowrate at t = 0.5 min, t = 1 min, t = 5 min.

Solution : The impulse response equation for tank is expressed as

$$Q_0(t) = \frac{A}{\tau} e^{-t/\tau}$$

Where,

$Q_0(t) = q_0(t) - q_{0s}$

$q_0(t)$ = outlet flowrate of liquid at time t = t

q_{0s} = steady state outlet flowrate

A = magnitude of impulse change

τ = time constant for tank

Magnitude of impulse change

$A = 0.05\ m^3/min$

The outlet flowrate of liquid is expressed as

$$q_0(t) = q_{0s} + \frac{A}{\tau} e^{-t/\tau}$$

$$q_0(t) = 0.6 + \left(\frac{0.05}{1}\right) e^{-t}$$

$$q_0(t) = 0.6 + 0.05\, e^{-t}$$

For time t = 0.5 min

$$q_0(t) = 0.6 + 0.05\ e^{-0.5}$$

$$q_0(t) = 0.63\ m^3/min$$

For time t = 1 min

$$q_0(t) = 0.6 + 0.05\ e^{-1}$$

$$q_0(t) = 0.618\ m^3/min$$

For time t = 5 min

$$q_0(t) = 0.6 + 0.05\ e^{-5}$$

$$q_0(t) = 0.6\ m^3/min$$

The outlet flowrate of liquid is

$$q_0(t) = 0.63\ m^3/min \text{ for } t = 0.5 \text{ min}$$

$$q_0(t) = 0.618\ m^3/min \text{ for } t = 1 \text{ min}$$

$$q_0(t) = 0.6\ m^3/min \text{ for } t = 5 \text{ min.}$$

Example 14 : A tank having cross sectional area of 0.5 m^2. The steady state flowrate of liquid is 0.8 m^3/min. The flowrate is subjected to linear change of 0.06 m^3/min. The time constant of the tank is 1 min. Determine the outlet flowrate of liquid at t = 1 min, t = 3 min and t = 5 min.

Solution : The response equation for first order system subjected to linear change is expressed as

$$Q_0(t) = A(t-\tau) + A\tau e^{-t/\tau}$$

Where,

$$Q_0(t) = q_0(t) - q_{0s}$$

$q_{0s}(t)$ = outlet flowrate of liquid at time t = t

q_{0s} = steady state outlet flowrate of liquid

A = slope of variation of linear change

The outlet flowrate of liquid is expressed as

$$q_0(t) = q_{0s} + A(t-\tau) + A\tau e^{-t/\tau}$$

For time t = 1 min

$$q_0(t) = 0.8 + 0.06\left[(1-1) + e^{-1}\right]$$

$$q_0(t) = 0.81 \ m^3/min$$

For time t = 3 min

$$q_0(t) = 0.8 + 0.06\left[(3-1) + e^{-3}\right]$$

$$q_0(t) = 0.9 \ m^3/min$$

For t = 5 min

$$q_0(t) = 0.8 + 0.06\left((5-1) + e^{-5}\right)$$

$$q_0(t) = 1.041 \ m^3/min$$

The outlet flowrate of liquid is

$$q_0(t) = 0.81 \ m^3/min \ \text{for } t = 1 \text{ min}$$

$$q_0(t) = 0.9 \ m^3/min \ \text{for } t = 3 \text{ min}$$

$$q_0(t) = 1.041 \ m^3/min \ \text{for } t = 5 \text{ min.}$$

Example 15 : A tank having cross sectional area of 0.3 m^2. The steady state flowrate of water is 0.6 m^3/min is subjected to a step change of magnitude 0.05 m^3/min. The time constant for the tank is 1 min. Determine the liquid level in the tank at t = 1 min.

Solution : The response equation for first order control system subjected to step change is expressed as

$$Q_0(t) = M(1-e^{-t/\tau})$$

Where,

$Q_0(t) = q_0(t) - q_{0s}$

$q_0(t)$ = the flowrate at outlet at time t = t

q_{0s} = steady state outlet flowrate

M = magnitude of step change

τ = time constant

Magnitude of step change

$M = 0.05\ m^3/min$

The liquid level in the tank is expressed as

$$q_0 = \frac{h}{R}$$

Where,

h = liquid level in the tank

R = resistance to flow of water

The time constant is expressed as

$\tau = A R$

Where,

A = cross - sectional area of tank.

$$R = \frac{\tau}{A}$$

$$R = \frac{1}{0.3} = 3.3$$

The outlet flowrate of water at t = 1 min

$$q_0(t) = q_{0s} + M(1-e^{-t/\tau})$$

$$q_0(t) = 0.6 + 0.05(1-e^{-1})$$

$$q_0(t) = 0.63\ m^3/min$$

The liquid level in the tank is expressed as

$h = q_0 R$

$h = 0.63 \times 3.3 = 1.9\ m$

The liquid level in the tank at time t = 1 min is 1.9 m

Example 16 : A pneumatic controller is used to control temperature range of 0 – 100°C for the output change from 30 to 90 kN/m² as the temperature changes from 90 to 105°C. Calculate the proportional band and sensitivity of the controller.

Solution : The proportional band of the controller is the error expressed as a percentage of the range of measured variable

$$\text{Proportional band} = \left(\frac{105-90}{100-0}\right) \times 100$$

$$= 15\%$$

The sensitivity of controller

$$K_C = \frac{\text{difference in pressure}}{\text{error}}$$

$$K_C = \left(\frac{90-30}{105-90}\right) = 4 \text{ kN/m}^2\ {}^0\text{C}$$

$\frac{60}{15}$ 4

Proportional band is 15%

Sensitivity of controller is 4 kN/m² °C

Example 17 : A thermometer having first order dynamics with time constant of 6 second. The thermometer is placed in a bath of temperature 30°C. The temperature of the bath is suddenly increased to 80°C. After 6 second the thermometer is subjected to 30°C. Calculate the temperature indicated by the thermometer at t = 3 second, t = 6 second and t = 10 second.

Solution : The response equation for first order control system subjected to step change is written as

$$Y(t) = M\ (1 - e^{-t/\tau})$$

Where,

C=

$$Y(t) = y(t) - y_s$$

y (t) = the temperature indicated by the thermometer at time t = t

y_s = the temperature indicated by the thermometer at steady state

M = magnitude of step change

The temperature indicated by the thermometer at t = 3 second is written as

$$y(t) = y_s + M(1 - e^{-t/\tau})$$

$$y(t) = 30 + 50\ (1 - e^{-3/6})$$

$$y(t) = 49.673^0\text{C}$$

For time t = 6 second

$$y(t) = 30 + 50\,(1 - e^{-6/6})$$

3

$$y(t) = 61.6^0\text{C}$$

After 6 second the thermometer is subjected to temperature of 30°C

Magnitude of step change

$$M = (30 - 61.6) = -31.6^0C$$

For time t = 10 second

$$y(t) = 61.6 - 31.6\ (1 - e^{-10/6})$$

$$y(t) = 35.9^0\ C$$

The temperature indicated by the thermometer

$y(t) = 49.673^0C$ at time t = 3 sec

$y(t) = 61.6^0C$ at time t = 6 sec

$y(t) = 35.9^0C$ at time t = 10 sec

Example 18 : A thermometer having first order dynamics is at steady state temperature of 30°C is subjected to impulse change of magnitude 50°C. The time constant for thermometer is 6 second. Calculate the temperature indicated by the thermometer at t = 3 second, t = 6 second and t = 18 second.

Solution : The response equation for first order control system subjected to impulse change is written as

$$Y(t) = \frac{A}{\tau} e^{-t/\tau}$$

Where,

$Y(t) = y(t) - y_s$

$y(t)$ = the temperature indicated by the thermometer at t = t

y_s = steady state temperature of the thermometer

A = magnitude of impulse change

τ = time constant of thermometer

The temperature indicated by the thermometer

$$y(t) = y_s + \frac{A}{\tau} e^{-t/\tau}$$

Magnitude of impulse change

$A = 50°C$

For time t = 3 second

$$y(t) = 30 + \frac{50}{6} e^{-3/6}$$

$$y(t) = 35^0C$$

For time t = 6 second.

$$y(t) = 30 + \frac{50}{6} e^{-6/6}$$

$$y(t) = 33^0C$$

For time t = 18 second

$$y(t) = 30 + \frac{50}{6} e^{-18/6}$$

$$= 30.41°C$$

The temperature indicated by the thermometer

$y(t) = 35°C$ at $t = 3$ second

$y(t) = 33°C$ at $t = 6$ second

$y(t) = 30.41°C$ at $t = 18$ second

Example 19 : A thermometer having first order dynamics with 1 mm diameter bulb the length of bulb is 6 mm. Specific heat of mercury is 1.38 kJ/kg°C. The heat transfer coefficient is 300 W /m^2°C. Calculate the time needed to attain 90 percent of the applied magnitude of the step change in temperature.

Density of mercury = 13500 kg/m^3.

Solution : The surface area of bulb of thermometer $A = \pi d l$

Where,

d = diameter of bulb

l = length of bulb

$$A = \pi d l$$

$$\therefore A = 3.14 \times 1 \times 10^{-3} \times 6 \times 10^{-3}$$

$$A = 18.8 \times 10^{-6} m^2$$

Mass of mercury in the bulb

$$m = \left(\frac{\pi d^2}{4}\right) l \rho$$

Where,

ρ = density of mercury

$$m = \frac{3.14 \times (10^{-3})^2}{4} \times 6 \times 10^{-3} \times 13500$$

$$m = 6.35 \times 10^{-5} kg$$

The time constant of thermometer

$$\tau = \frac{m\,C_p}{h\,A}$$

$$\tau = \frac{6.35\times10^{-5}\times1.38\times10^{3}}{300\times18.8\times10^{-6}}$$

$\tau = 15.5$ sec

The response equation for first order control system subjected to step change is expressed as

$$Y(t) = M\,(1-e^{-t/\tau})$$

The time needed to attain 90 percent of the magnitude of step change

$$Y(t) = 0.9\,M$$

Magnitude of step change = M

$$0.9\,M = M\,(1-e^{-t/15.5})$$

$$e^{-t/15.5} = (1-0.9) = 0.1$$

$t = 35.6$ sec.

Example 20 : A tank having time constant of 1 min and area of tank is 10 m^2. The steady state inlet flowrate is 5 m^3/min. The flowrate is suddenly increased to 60 m^3/min for 0.1 min by adding 5.5 m^3 of water. Calculate the liquid level at t = 0.5 min.

Solution : The transfer function for the tank is expressed as

$$\frac{H(s)}{Q_i(s)} = \frac{R}{(\tau s+1)}$$

The time constant for the tank

$\tau = A\,R$

A = cross sectional area of tank

R = resistance to flowrate

$$R = \frac{\tau}{A} = \frac{1}{10} = 0.1$$

Magnitude of step change

$$M = (60-5) = 55\,m^3/\text{min}$$

$$Q_i(t) = 55\,[u(t) - u(t-0.1)]$$

$$Q_i(s) = \frac{55}{s}(1-e^{-0.1s})$$

$$\frac{H(s)}{Q_i(s)} = \frac{R}{\tau s+1}$$

$$\frac{H(s)}{Q_i(s)} = \left(\frac{0.1}{s+1}\right)$$

$$H(s) = \left(\frac{55\times 0.1}{s}\right)\left(\frac{1-e^{-0.1s}}{s+1}\right)$$

$$H(s) = 5.5\left[\frac{1}{s(s+1)} - \frac{e^{-0.1s}}{s(s+1)}\right]$$

$$H(t) = 5.5\left[(1-e^{-t}) - (1-e^{-(t-0.1)})\right]$$

The liquid level in the tank for time t = 0.5 min

$$H(t) = 5.5\left[(1-e^{-0.5})-(1-e^{-(0.5-0.1)})\right]$$

H (t) = 0.35 m.

Example 21 : An aqueous solution in a tank is heated by a coil. The density and the specific heat of solution are 1000 kg/m^3 and 4 k J/kg°C respectively. The steady state temperature of the tank content is 30°C. the feed rate is 1.5 m^3/min and the volume in of the tank is 1.5 m^3. The heating coil is subjected to a step change of 500 kW. Calculate the outlet temperature of the solution for t = 0.5 min, t = 1 min, t = 3.5 min, t = 5 min and t = 6 min.

Solution : The transfer function for the tank in which fluid is heated is expressed as

$$\frac{Y(s)}{X(s)} = \frac{1}{\tau s+1}$$

Where,

$$\tau = \frac{m\,C_P}{U\,A}$$

The heat load to the tank is expressed as

$$Q = U A \Delta T = m\,C_P \frac{dT}{dt}$$

Where,

Q = heat load to the tank in kW

C_p = specific heat of solution k J/kg°C

U = overall heat transfer coefficient kW/m^2 °C

m = mass of solution kg.

$$Q = U A \Delta T = m_1\, C_p\, d\,T$$

$$UA = \frac{Q}{\Delta T}$$

m_1 = mass flowrate of solution $\frac{kg}{s}$,

The time constant of tank is expressed as

$$\tau = \frac{m\,C_p}{U\,A} = \frac{m\,C_p}{(Q/\Delta T)}$$

Mass of liquid in the tank

m = density of solution × volume of solution

$m = 1000 \times 1.5 = 1500$ kg

The time constant is expressed as

$$\tau = \frac{m\,C_p}{(Q/\Delta T)}$$

Heat load per unit change in temperature is

$$\frac{Q}{\Delta T} = m_1\,C_p$$

$$\tau = \frac{m\,C_p}{(Q/\Delta T)} = \frac{m\,C_p}{m_1\,C_p} = \frac{m}{m_1}$$

$$\tau = \frac{\text{Mass of solution}}{\text{Mass flowrate of solution}}$$

$$\tau = \frac{\text{Volume of solution}}{\text{Volumetric flowrate of solution}}$$

$$\frac{Q}{\Delta T} = \frac{1.5 \times 1000 \times 4}{60} = 100\ \text{kW}/^0\text{C}$$

$$\tau = \frac{m\,C_p}{(Q/\Delta T)} = \frac{1500 \times 4}{100} = 60\ \text{sec}$$

$\tau = 1$ min

The transfer function of the heating tank is expressed as

$$\frac{Y(s)}{X(s)} = \frac{1}{\tau s + 1}$$

$$X(s) = \frac{Q(s)}{m_i C_P}$$

$$\frac{Y(s)}{Q(s)} = \left(\frac{1}{m_1 C_P}\right)\left(\frac{1}{\tau s+1}\right)$$

The heating coil is subjected to step change of 500 kW

$$Q(t) = 500 \text{ kW}$$

$$Q(s) = \frac{500}{s}$$

$$Y(s) = \left(\frac{500}{m_1 C_P}\right)\left(\frac{1}{s}\right)\left(\frac{1}{\tau s+1}\right)$$

$$Y(t) = \left(\frac{500}{100}\right)(1-e^{-t/\tau})$$

Where,

$Y(t) = y(t) - y_s$

$y(t)$ = the temperature of solution at outlet at t = t

y_s = steady state temperature of solution at outlet

The temperature of solution at the outlet is expressed as

$$y(t) = y_s + 5(1-e^{-t/\tau})$$

For $\tau = 1$ min, $y_s = 30°C$

$$y(t) = 30 + 5(1-e^{-t})$$

For time t = 0.5 min

$$y(t) = 30+5(1-e^{-0.5}) = 31.9^0C$$

For time t = 1 min

$$y(t) = 30+5(1-e^{-1}) = 31.16^0C$$

For time t = 3.5 min

$$y(t) = 30+5(1-e^{-3.5}) = 34.849^0C$$

For time t = 5 min

$$y(t) = 30 + 5(1-e^{-5}) = 34.9^0C$$

For time t = 6 min

$$y(t) = 30 + 5(1-e^{-6}) = 34.98^0C$$

$$X(s) = \frac{Q(s)}{m_1 C_P}$$

$$\frac{Y(s)}{Q(s)} = \left(\frac{1}{m_1 C_P}\right)\left(\frac{1}{\tau s+1}\right)$$

The heating coil is subjected to step change of 500 kW

$Q(t) = 500$ kW

$$Q(s) = \frac{500}{s}$$

$$Y(s) = \left(\frac{500}{m_1 C_P}\right)\left(\frac{1}{s}\right)\left(\frac{1}{\tau s+1}\right)$$

$$Y(t) = \left(\frac{500}{100}\right)(1-e^{-t/\tau})$$

Where,

$Y(t) = y(t) - y_s$

$y(t)$ = the temperature of solution at outlet at $t = t$

y_s = steady state temperature of solution at outlet

The temperature of solution at the outlet is expressed as

$$y(t) = y_s + 5(1-e^{-t/\tau})$$

For $\tau = 1$ min, $y_s = 30$°C

$$y(t) = 30 + 5(1-e^{-t})$$

For time $t = 0.5$ min

$$y(t) = 30+5(1-e^{-0.5}) = 31.9^0C$$

For time $t = 1$ min

$$y(t) = 30+5(1-e^{-1}) = 31.16^0C$$

For time $t = 3.5$ min

$$y(t) = 30+5(1-e^{-3.5}) = 34.849^0C$$

For time $t = 5$ min

$$y(t) = 30 + 5(1-e^{-5}) = 34.9^0C$$

For time $t = 6$ min

$$y(t) = 30 + 5(1-e^{-6}) = 34.98^0C$$

The temperature of the solution at outlet is

$y(t) = 31.9^0C$ at $t = 0.5$ min

$y(t) = 31.16^0C$ at $t = 1$ min

$y(t) = 34.849^0C$ at $t = 3.5$ min

$y(t) = 34.9^0C$ at $t = 5$ min

$y(t) = 34.98^0C$ at $t = 6$ min.

Example 22 : An aqueous solution in a tank is heated by a coil. The density and the specific heat of solution are 1000 kg/m^3 and 4 k J/kg °C respectively. The steady state temperature of the tank content is 30°C. The feed rate is 1.5 m^3/min and the volume of the tank is 1.5 m^3. The power supplied to the coil is suddenly increased by 800 kW. After 1 min the power supplied is decreased to 500 kW. Calculate the outlet temperature of the solution for, t = 1.8 min, t = 3.5 min and t = 6.5 min.

Solution : The transfer function for the heating tank is expressed as

$$\frac{Y(s)}{X(s)} = \frac{1}{\tau s+1}$$

The heat load to the tank is expressed as

$$Q = U A \Delta T = m_1 C_p dT$$

The time constant is expressed as

$$\tau = \frac{m C_P}{U A}$$

Where,

Q = heat load to the tank in kW

C_p = specific heat of the solution k J/kg °C

U = overall heat transfer coefficient kW/m^2 °C

m = mass of solution kg

$$\tau = \frac{m C_P}{U A} = \frac{m C_P}{(Q/\Delta T)}$$

$$\tau = \frac{m C_P}{m_1 C_P} = \frac{m}{m_1}$$

$$\tau = \frac{\text{Mass of solution}}{\text{Mass flowrate of solution}}$$